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FUNDAMENTAL PRINCIPLES OF MECHANICS
ANALYTIC STATICS
KINEMATICS AND KINETICS
HYDROSTATICS
PNEUMATICS
RUDIMENTS OF ANALYTIC GEOMETRY

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PREFACE

The volumes of the International Library of Technology are made up of Instruction Papers, or Sections, comprising the various courses of instruction for students of the International Correspondence Schools. The original manuscripts are prepared by persons thoroughly qualified both technically and by experience to write with authority, and in many cases they are regularly employed elsewhere in practical work as experts. The manuscripts are then carefully edited to make them suitable for correspondence instruction. The Instruction Papers are written clearly and in the simplest language possible, so as to make them readily understood by all students. Necessary technical expressions are clearly explained when introduced.

The great majority of our students wish to prepare themselves for advancement in their vocations or to qualify for more congenial occupations. Usually they are employed and able to devote only a few hours a day to study. Therefore every effort must be made to give them practical and accurate information in clear and concise form and to make this information include all of the essentials but none of the non-essentials. To make the text clear, illustrations are used freely. These illustrations are especially made by our own Illustrating Department in order to adapt them fully to the requirements of the text.

In the table of contents that immediately follows are given the titles of the Sections included in this volume, and under each title are listed the main topics discussed.

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CONTENTS

NOTE.—This volume is made up of a number of separate Sections, the page numbers of which usually begin with 1. To enable the reader to distinguish between the different Sections, each one is designated by a number preceded by a Section mark (§), which appears at the top of each page, opposite the page number. In this list of contents, the Section number is given following the title of the Section, and under each title appears a full synopsis of the subjects treated. This table of contents will enable the reader to find readily any topic covered.

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KINEMATICS AND KINETICS, § 30 *Pages*

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RUDIMENTS OF ANALYTIC GEOMETRY

(*Continued*)

FUNDAMENTAL PRINCIPLES OF MECHANICS

MOTION

INTRODUCTORY DEFINITIONS

MATTER—BODY

1. Matter may be defined as either: (*a*) whatever is known, or can be supposed capable of being known, through the sense of touch; or (*b*) that which occupies space

2. A **phenomenon** is whatever happens in nature. The motions of the heavenly bodies, the rolling of a ball on the ground, the generation of steam from water, the formation of rust on the surface of iron, the circulation of the blood, the beating of the heart, thinking, speaking, walking—all these are phenomena It will be seen from the definition and examples just given that the word phenomenon, as understood and used in science, does not mean "something extraordinary or wonderful." For science, there is nothing extraordinary, and a phenomenon is simply a fact—anything that happens.

3. Body, Substance, Material.—A **body** is any limited portion of matter, as a block of wood, a coin, a stone, a piece of flesh, an animal. Another meaning of the word will be given presently.

4. All bodies do not possess the same properties, or do not possess them in the same degree. This fact makes it necessary to distinguish different kinds of matter, to which different names are given: as, *iron*, *water*, *air*, *flesh*.

The word **substance** is often employed in the same sense as the word *matter,* but in its more common use, it refers to matter of a special kind. Thus, water, steel, iron, are substances, or kinds of matter.

The word *body* is also used, especially in physics and chemistry, to denote matter of a particular kind. In this sense, water, air, hydrogen, gold, are called bodies. In mechanics, however, the term is usually taken in the sense given to it in Art. **3.**

5. Those substances that are used for works of art are called **materials:** steel, iron, brick, stone are examples.

6. Particle.—A **particle** is a body so small that its dimensions may be disregarded. A particle is accordingly treated as if it were a geometrical point, except that it is sometimes necessary to consider some of its physical properties, such as weight. A particle is often called a **material point.** All bodies are treated as being composed of an indefinitely large number of particles.

7. It will be noticed in what follows that, in many cases, properties and principles that have been established and stated as relating to material points are extended to whole bodies. Thus, after investigating and explaining the laws governing the motion of a material point under certain circumstances, these laws are illustrated by taking as examples the motions of wheels, cars, engines, etc. The reason is that in cases of this kind the properties considered do not depend on the size of the bodies. For example, the speed of a car moving in a straight track is measured by the speed of any of its points, and what applies to the speed of a point applies to the speed of the car.

So, too, the force requi[illegible] to move in a straight track a train of a certain weig[illegible] the same as the force necessary to move a particle of the same weight; and, therefore, the

laws obtained in the case of a particle may be applied to the motion of a train or any other body, so long as the only thing to be taken into account is the weight of the body.

8. Deformation.—When a body is pulled, pressed, or struck in any direction, its shape is more or less changed, either permanently or temporarily. A change of this kind is called **deformation**, and the body undergoing it is said to be **deformed**.

9. A **rigid body** is a body that is not deformable, that is, a body whose form and dimensions remain the same, to whatever action it may be subjected

There are no absolutely rigid bodies in nature; but it often simplifies matters to consider bodies at first as being perfectly rigid, which is equivalent to neglecting their deformability, the latter being afterwards taken into account. Besides, in many cases, deformability plays so small a part that, for practical purposes, it may be neglected entirely.

MOTION AND REST

10. Relative Position.—The **relative position** of a point with respect to another is determined by the length and direction of the straight line between the two points When either the length or the direction, or both the length and direction, of this line change, the relative position of the two points is said to change

The relative position of two bodies with respect to each other is determined by the relative position of the points of one body with respect to those of the other. When the relative position of one or more points in one of the bodies with respect to the points of the other body changes, the relative position of the two bodies, with respect to each other, is also said to change.

11. Motion is a change in the relative position of two bodies, and is said to be possessed by either body with respect to the other.

12. **Rest** is the condition of two bodies not in motion with respect to each other, and is said to belong to either body with respect to the other. Each body is said to be **fixed** with respect to the other body.

13. When the motion or rest of a body is referred to, without specifying any other body, it is generally understood that the other body is the earth or its surface. Thus, the motion of a train, of a steamer, of a horse, as usually spoken of, means the change of position of the train, steamer, or horse with respect to the ground—that is, to the surface of the earth, or to objects on that surface.

14. A body may be in motion with respect to another body, and at rest with respect to a third body. For example, the smokestack of a locomotive is at rest with respect to the boiler, since their relative position does not change; yet, both may be moving with respect to objects on the ground. Likewise, a man standing on the deck of a moving vessel is at rest with respect to the vessel, but in motion with respect to the water, the shore, etc.

15. **Path.**—The **path**, or **trajectory**, of a moving point is the line over which the point moves, or, in geometrical language, the line generated by the moving point. The path of a point may be straight or curved, or a combination of straight and curved lines.

16. **Initial and Final Position.**—When the motion of a point during a certain time or over a certain portion of its path is considered, the position occupied by the point at the beginning of the time is called the **initial position** of the point, and the position at the end of the time is called the **final position**. Suppose that a particle starts from A, Fig. 1, and moves so that it describes, in a certain time, the curve AMB; this curve is the path of the particle. With respect to the time considered, A is the initial and B the final position of the particle. . . .

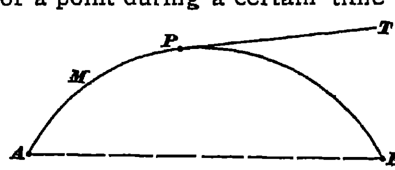


FIG. 1

17. Displacement.—The length of the straight line joining the initial and the final position of a moving point is called the **displacement** of the point while the point passes from the former to the latter position. In Fig 1, in which the point moves from A to B in the curved path AMB, its displacement during this motion is the length of the straight line AB

18. The **space** passed over, or described, in a certain time by a moving point is the length of the part of the path over which the body passes during that time. In Fig 1, the space described by the particle between A and B is the length of the curve AMB

19. Direction of Motion in a Curve.—By the direction of a curve at any point is meant the direction of its tangent at that point So, too, when a point is moving in a curved path, the direction of motion at any moment is the direction of the tangent to the path at the point occupied at that moment by the moving point. In Fig 1, the line PT just touches the curve AMB at P, it is, therefore, a tangent to the curve at P, and shows the direction in which the particle is moving when it reaches the position P

VELOCITY

VELOCITY IN UNIFORM MOTION

20. Uniform Motion.—A point is said to move with **uniform motion** when it passes over equal distances in any and every two equal intervals of time The fact, for instance, that a point moves over 10 feet during the first second of its motion, and over an equal space during the fiftieth second, is not enough to define the motion as uniform: that the motion may be uniform, the point must describe the same space (10 feet in this case) in every second, whatever instant is chosen in which to begin to count the time; thus, the space described by the point between the middle of the third second and the middle of

the fourth must be the same as the space described in the twentieth second, or during the second in which the time of motion changes from 27 15 to 28.15 seconds, or, in short, during any and every interval of 1 second.

21. Definition of Velocity.—It is a familiar fact that bodies move more or less rapidly, more or less slowly By this is meant that they require more or less time to move from one place to another, or that they move over longer or shorter distances in a given time. In ordinary language, the term *speed* is used to denote the degree of quickness or rapidity of motion. The measure of the speed, or of the degree of quickness, of a motion is called **velocity.** How this measure is obtained will now be explained.

22. Uniform Velocity.—Let a point M, Fig. 2, move along the path $A\,B$, which may have any form, either curved or straight; and suppose the motion to be such that equal spaces are passed over in equal times; that is, suppose the motion to be uniform. Also, let the number of units of length passed over in a unit of time (as the number of feet passed over in 1 second) be denoted by v. It is evident that the magnitude of this number v depends on how fast the point is moving. Thus, if the point passes over 10 feet in 1 second, it is moving twice as fast as if it passed over only 5 feet in 1 second Hence, the number v may be used to measure the rapidity of the motion, in which case v is the velocity of the moving point. In general terms, then, the **velocity** of a point moving with uniform motion is the space passed over by the point in a unit of time.

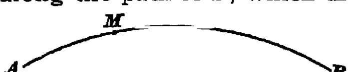


FIG 2

When, as here assumed, the motion is uniform, the velocity also is said to be **uniform, or constant.** What is meant by variable motion and variable velocity will be explained presently.

Velocity is expressed in units of length per unit of time. For example, if a body moving uniformly passes over 7 feet in 1 second, its velocity is 7 feet per second. In 1 minute, the body passes over a space of $7 \times 60 = 420$ feet; hence,

its velocity can be expressed also as 420 feet per minute. A train that moves uniformly and travels 45 miles an hour has a velocity of 45 miles per hour.

23. Formulas for Uniform Motion.—Let a point or body moving uniformly pass over a space s during any time t. Then, it is obvious that the space described in a unit of time is $s \div t$. Hence,

$$v = \frac{s}{t} \qquad (1)$$

For example, if the point moves over a space of 10 feet in 2 seconds, $v = 10 \div 2 = 5$ feet per second

If the velocity and the time are known, the preceding formula gives the space passed over.

$$s = v\,t \qquad (2)$$

Similarly, if the velocity and the space are given, the same formula, or formula **2**, gives the time.

$$t = \frac{s}{v} \qquad (3)$$

EXAMPLE 1 —The wheels of a carriage moving with uniform velocity make 3,760 revolutions between two stations A and B The diameter of each wheel is 4 feet, and the time employed by the carriage to travel from one station to the other is 7 5 hours It is required to find the velocity of the carriage, in feet per second (By the velocity of the carriage is meant the velocity of any of its points)

SOLUTION —During each revolution, the carriage passes over a distance equal to the circumference of the wheel, or $4 \times 3\ 1416$ ft. The total distance traveled, or space passed over, in feet, is $s = 3{,}760 \times 4 \times 3.1416$ The time required to travel this distance is 7 5 hr. As the velocity is required in feet per second, this time must be reduced to seconds, which gives $t = 7\ 5 \times 60 \times 60$ Substituting in formula **1** the values of s and t just found,

$$v = \frac{3{,}760 \times 4 \times 3\ 1416}{7\ 5 \times 60 \times 60} = 1.75 \text{ ft per sec} \quad \text{Ans}$$

EXAMPLE 2.—The distance between two ports is 400 miles What must be the velocity, in knots, of a vessel that will cover the distance in 18 hours? (A knot is a velocity of 6,080 feet per hour.)

SOLUTION —To find the velocity in feet per hour, we have, $s = 400$ mi. $= (400 \times 5{,}280)$ ft , $t = 18$ hr Therefore, by formula **1**,

$$v = \frac{400 \times 5{,}280}{18} \text{ ft. per hr.} = \frac{400 \times 5{,}280}{18 \times 6{,}080} \text{ knots} = 19.298 \text{ knots. } \textbf{Ans.}$$

VELOCITY IN VARIABLE MOTION

24. Variable motion is that motion in which the moving point, or body, passes over unequal spaces in equal times, or over equal spaces in unequal times

25. Average or Mean Velocity.—It will be remembered that the distinguishing characteristic of uniform motion is the equality of all distances passed over in equal intervals of time, and that the quotient obtained by dividing any space s by the number t of units in the corresponding interval of time is a constant quantity (v, formula **1** of Art **23**). Unless all these quotients are equal, the motion, and therefore the velocity, is not uniform

Suppose, for example, that a train moves over a distance of 60 miles in 45 minutes. If it is known beforehand that the motion is uniform, the velocity may be found by dividing 60 miles by 45, which gives $v = \frac{60}{45} = \frac{4}{3}$ miles per minute. In this case, the space described in any number t of minutes would be, by formula **2** of Art. **23**, $v\,t = \frac{4}{3}t$ miles.

Suppose, on the contrary, that it is not known whether the motion has been uniform during the 45 minutes, but that the distance passed over during the first 5 minutes is known to be 5.5 miles, and that the distance passed over between the end of the thirtieth and the end of the fortieth minute is 13.98 miles Then, the quotients obtained by dividing the spaces by the numbers measuring the corresponding times are, respectively,

$$\frac{5.5}{5} = 1.10;\ \frac{13.98}{10} = 1.398;\ \frac{60}{45} = \frac{4}{3}$$

Since these quotients are not equal, there is no uniform motion, and therefore no uniform velocity.

If, however, another train is imagined moving with a uniform velocity of $\frac{4}{3}$ miles per minute, this train will pass over the distance of 60 miles in 45 minutes. For this reason, $\frac{4}{3}$ miles per minute is called **mean, or average, velocity** of the actual train considered; it is not a velocity that

the train really possesses, but rather the equivalent velocity with which a body moving uniformly would move over the same space in the same time Likewise, if the motion of the train during the first 5 minutes is considered, its mean velocity, for that interval, is $\frac{5.5}{5} = 1\ 10$ miles per minute. In general, if a body moves over a space s in the time t, its mean velocity v_m, during that time, is defined mathematically by the formula

$$v_m = \frac{s}{t}$$

26. Variable Velocity—Instantaneous Velocity. When a point is moving with variable motion, its velocity at any instant is the velocity that the point would have if, at that instant, its speed ceased to change, that is, if, after that instant, the point moved neither faster nor more slowly than it is moving. Since the motion is variable, the velocity at any instant is not any velocity with which the body actually moves over any part of its path, but simply the velocity with which it would move if its speed became invariable.

The velocity of a moving point or body at any given moment is called the **instantaneous velocity** at that moment. In variable motion, the velocity is said to be **variable**, because it changes or varies from instant to instant.

27. To illustrate the character of variable motion and velocity, imagine a train A to start from rest and leave a station at the same time that another train B is passing the station with a uniform velocity of 60 miles per hour, and suppose the two trains to run on two parallel tracks. At first, B will move ahead of A, since A starts from rest. Suppose, however, that A moves faster and faster, so that to a person in B the motion of B with respect to A will appear to become slower and slower; there may then be a time when B will be losing instead of gaining speed, with respect to A, or when A will appear to be moving past B. Thus, during the first minute, B will move over 1 mile, and

A may move over only $\frac{3}{4}$ mile; but, during the following minute, A may move over $1\frac{1}{2}$ miles, and then it will be $\frac{1}{4}$ mile ahead of B. Since A is at first losing space with respect to B and then is gaining, there must be an instant at which A is neither losing nor gaining; at that instant the two trains are evidently at rest with respect to each other, and if the speed of A ceased to change, A would continue to move with the same velocity as B, or 60 miles per hour. The velocity of A at that instant is, therefore, 60 miles per hour.

It must be understood that, as already stated, the train A does not actually move with the velocity of 60 miles per hour during any interval of time; but this is the velocity with which A would move if, after the instant considered, its motion underwent no further change.

28. Direction and Magnitude of Velocity.—By the **direction** of the velocity of a moving point is meant the direction in which the point is moving.

29. By the **magnitude** of a velocity is meant the numerical value of the velocity, expressed in units of length per unit of time, as feet per second or miles per hour.

EXAMPLE —A float moves down a stream from a point A to a point D, a distance of 800 feet, in $2\frac{1}{2}$ minutes The float is observed at two intermediate points B and C, whose distances from A are, respectively, 350 and 575 feet, and it is found that it moves from A to B in 45 seconds, and from B to C in 70 seconds. Required, the mean velocity, in feet per second, of the surface of the stream (*a*) between A and D, (*b*) between A and B, (*c*) between B and C; (*d*) between C and D

SOLUTION —(*a*) Here s = 800 ft , t = $2\frac{1}{2}$ min = 150 sec ; and the formula of Art 25 gives

$$v_m = \frac{s}{t} = \frac{800}{150} = 5.33 \text{ ft. per sec. Ans.}$$

(*b*) Here s = 350 ft , t = 45 sec., and, therefore,

$$v_m = \frac{s}{t} = \frac{350}{45} = 7\ 78 \text{ ft per sec. Ans.}$$

(*c*) Here s = 575 ft − 350 ft = 225 ft., t = 70 sec , and, therefore,

$$v_m = \frac{s}{t} = \frac{225}{70} = 3.214 \text{ ft. per sec Ans}$$

(d) Here $s = 800$ ft $- 575$ ft $= 225$ ft, $t = 150$ sec $- (45 + 70)$ sec. $= 35$ sec, and, therefore,

$$v_m = \frac{s}{t} = \frac{225}{35} = 6\,429 \text{ ft per sec. Ans.}$$

EXAMPLES FOR PRACTICE

1 A train moves with uniform velocity between two stations 375 miles apart, the distance is traveled in 7½ hours. What is the velocity of the train, in feet per second? Ans 73 333 ft per sec

2 A train moves from station A to station B, a distance of 90 miles, in 1⅔ hours, from station B to station C, a distance of 17 miles, in ½ hour; and from station C to station D, a distance of 64 miles, in 2 hours Find its mean velocity, in feet per second (a) between stations A and D, (b) between stations B and C; (c) between stations B and D.

Ans $\begin{cases} (a)\ 59\,012 \text{ ft per sec.} \\ (b)\ 49\,867 \text{ ft per sec} \\ (c)\ 47\,52 \text{ ft. per sec.} \end{cases}$

FORCE AND MASS

FORCE

DEFINITIONS RELATING TO FORCE—MEASURE OF FORCE

30. Definition of Force.—It is known from experience that, when a body is at rest, it can be set in motion by the action of another body. Thus, if a block of iron is lying on the ground, it can be set in motion by pulling it with a rope, by pushing it with the hand, or by placing a strong magnet near it. Furthermore, the block may be pushed by one person in one direction, while another person pushes it in the opposite direction; or, the magnet may be placed on one side of the block, at the same time that the block is pulled by a string from the opposite side. In cases of this kind, there may be no motion, on account of the neutralizing effects of two or more actions on the same body. We say, however, that there is a **tendency** to motion; for the moment one of the opposing actions is removed, the other causes the body to move

31. This action of one body on another, producing, or tending to produce, motion in the latter body, is called **force**, and the former body is said to **exert force** on the latter

The characteristic of the action known as force is, then, that, when exerted on a body originally at rest, and not influenced by other bodies, it results in the motion of the first-mentioned body. The nature of the ensuing motion, and the result of the action in question, when the body acted on is already in motion or under the action of other bodies, are complicated phenomena to be either actually determined by experiment or inferred from experimental data.

32. Balanced and Unbalanced Forces.—If two or more forces act on a body in such a manner that they cause no motion, owing to the neutralizing effects they tend to produce, each force is said to be **balanced** by the combined action of the others, and is referred to as a **balanced force.**

When a force acts on a body, and there is no opposing force preventing the motion of the body, the force is called an **unbalanced force.**

These definitions apply to moving bodies as well as to bodies at rest If two or more forces are balanced when exerted on a body at rest, they are also balanced when the body is in motion; that is, they do not affect the motion of the body.

33. Equilibrium.—A body is in **equilibrium** when it is under the action of balanced forces. The balanced forces themselves are also said to be in equilibrium.

It should be noticed that equilibrium and rest are not equivalent terms A body may move uniformly while acted on by balanced forces, in which case it is in equilibrium, but not at rest In this case, however, the motion of the body is not due to the balanced forces acting on it. This subject will be better understood after the law of inertia, presently to be explained, has been studied

34. Weight.—Experience teaches that, when a body is unsupported, it falls to the ground. This fact is ascribed to a force exerted by the earth on all bodies, which force is known

by the general name of **force of attraction**, or simply **attraction**, and also **force of gravity**, or simply **gravity**.

The attraction of the earth on any particular body is called the **weight** of the body The methods of comparing the weights of bodies are well known.

35. Measure of Force.—For engineering purposes, force is expressed in units of weight, such as pounds, kilograms, etc. The reason for this will be more apparent when it is considered that every force can be replaced by a weight. Thus, suppose a body P, Fig. 3, to be suspended from the extremity A of a perfectly symmetrical beam $A B$ resting at its center on a knife edge F. If we wish to prevent P from falling, we may pull downwards on the string at Q, or attach at Q a weight equal to P, or tie a small piece of iron at Q and place a magnet underneath. In all these cases, the weight of P balances, and is therefore equivalent to, the force at Q, by whatever means the latter may be produced.

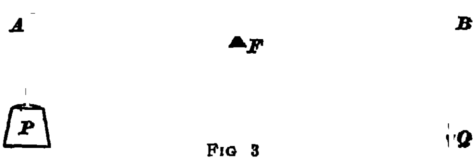


FIG 3

As the two forces are equivalent, the one may be measured by the other, and we may say that the force acting at Q is 20 pounds, kilograms, tons, etc., according as P weighs 20 pounds, kilograms, tons, etc. Suppose, for instance, that the weight of P is 10 pounds, and that we pull at Q with sufficient force to keep P from falling, but without moving it. Then, the force with which we are pulling is 10 pounds. If P falls, the pull is less than 10 pounds; and if it rises, the pull is greater than 10 pounds.

36. Magnitude of a Force.—By the **magnitude** of a force is meant the numerical value of the force, expressed in units of weight. If, for example, a force is equivalent to a weight of 12 pounds, its magnitude is 12 pounds.

37. Direction of a Force.—The **direction** of a force is the direction in which the force moves, or tends to move, the body on which it acts.

DEFINITION AND DIVISIONS OF MECHANICS

38. Mechanics is the science of force and motion. This science is divided into two general branches. (*a*) *dynamics*, (*b*) *kinematics* or *phoronomics*.

39. Dynamics is the science of force and its effects. In this science, the forces applied to bodies are given, and the resulting effects of these forces are determined, or the conditions of motion are given, and the forces necessary to produce that motion determined. Rest is considered as a special case of motion in which the velocity is zero.

40. Kinematics treats of motion alone, without reference to either force or the physical or mechanical properties of bodies. Suppose, for instance, that a body is moving in a circle with uniform velocity, and that it is required to determine what force is necessary to preserve this kind of motion, this is a problem in dynamics, for it relates to force Suppose that it is required to determine what space the body travels in a certain time; this is a problem in kinematics, for it can be solved without having regard to anything but the velocity and path of the body.

41. Subdivisions of Dynamics.—Dynamics is subdivided into two branches: (*a*) *kinetics*, (*b*) *statics*.

42. Kinetics treats of unbalanced forces and their effects.

43. Statics treats of the equivalence and equilibrium of forces.

NOTE —The preceding divisions and definitions are of very recent origin Formerly, the term *dynamics* was used in the sense in which *kinetics* is now used, the latter term being then unknown Even today, the old meaning is frequently given to the term *dynamics*, the best modern writers, however, use it in the sense just defined —that is, to denote the science of force in general, whether it produces motion or not

44. Applied mechanics is the application of mechanical principles to the works of human art.

FUNDAMENTAL LAWS OF DYNAMICS

LAW OF INERTIA

45. Statement of the Law.—In giving the definition of force (Art. **30**), reference was made to the fact, known from experience, that a body may act on another body, originally at rest, in such a manner as to produce motion in the latter body. This action was defined as force. That definition relates only to the effect of force on a body originally at rest. The law presently to be stated expresses the general effect of force under any circumstances; that is, whether the body on which it is exerted is at rest or in motion. This law, known as the **law of inertia**, or the **first law of motion**, may be formulated as follows:

If a body is not acted on by force, the body, as a whole, is either at rest or moving uniformly in a straight path.

This is not as definite a statement of the law of inertia as can be given. But the complete statement requires, in order to be understood, some familiarity with mechanical principles and conceptions of a high order, with which the beginner cannot be supposed to be acquainted

By saying that a body as a whole is in motion, it is meant that all the points of the body are in motion. Each of the wheels of a locomotive, for instance, has a motion as a whole, and, if the track is straight, the wheel, as a whole, is said to be moving in a straight path, whose direction is that of the track. Any one would understand what was meant by saying that a billiard ball was rolling in a direction parallel to one of the cushions of the billiard table.

In the simple cases of motion just given, the direction of the motion of the body as a whole is plainly seen to be the direction in which its center of figure is moving. In the case of unsymmetrical bodies, the direction of their motion is the direction in which a certain point in them, called the **center of gravity**, is moving. For the present, however, the student may think of the motion of symmetrical bodies

only, such as balls, disks, cubes, prisms, etc., all of which have a center of figure.

46. Facts on Which the Law is Founded.—That a body does not start to move by itself is a fact so familiar that it is scarcely necessary to state it That a body, once set in motion, does not come to rest unless acted on by force is not so obvious. Bodies often come to rest apparently by themselves, without any assignable cause, and this gave rise to the belief, prevalent among the ancients, and still entertained by some, that rest is the natural condition of all bodies, and that all moving bodies have a tendency to come to rest. A little consideration will show that this is an erroneous view of the nature of motion. If a block is placed on a stone pavement and struck sidewise, it will slide for a short distance and soon come to rest; if the experiment is tried on an asphalt pavement, the block will move farther and come to rest after a longer time; it will slide longer and farther on a piece of marble, and longer and farther still on a sheet of ice. These facts plainly indicate that the tendency to come to rest is not inherent in the block itself, but depends on the action of other bodies—namely, the bodies on the surfaces of which the block slides, and we naturally infer that, were it not for this action (or **frictional resistance**, as it is called), the block would not come to rest at all.

Another common obstacle to motion is the **resistance of the medium**, by which is meant the action of the substance through which the body moves. Thus, if a block is placed on a horizontal surface, immersed in mercury, and then struck sidewise, it will move through a very short distance. If the experiment is repeated in water, the block will describe a longer space before coming to rest. The space will be longer in air, and longer still in a space (called a **vacuum**) from which the air has been removed by means of an air pump. Here, as before, the inference is that, did the medium offer no resistance, or, more properly, were there no medium, the body would move forever in a straight path.

47. In the preceding examples, it will be observed that, whether the body is brought to rest by frictional resistance or by the resistance of the medium, the diminution of its velocity is continuous, in other words, the body comes to rest gradually, and its velocity constantly diminishes while the resisting force acts

An example of the action of force in increasing the velocity of a moving body is afforded by the familiar phenomenon of a falling body Here, the body is constantly acted on by the attraction of the earth, and its velocity is observed to increase very rapidly As another example, suppose that a person pushes a car on a horizontal surface. The car moves at first with almost no velocity; but, as the person continues to push, the velocity keeps growing greater and greater.

48. The necessity of applying force to a moving body in order to change the direction of its motion is also a familiar fact. If a prismatic block is sliding on a surface in the direction of its axis, no change in the direction of its motion will be observed unless the prism is struck, or pulled, or pushed, or in some other way interfered with by some other body. Of course, all forces do not change the direction of motion, but only those exerted by bodies that are, so to speak, outside the path of the moving body. Thus, in the example just given, a pull exerted by a rope in the direction of the axis of the prism would cause no change in the direction of motion; but, if the rope were pulled at right angles to the axis of the prism, the direction of motion would evidently change.

49. Summing up, it may be said that, *whenever the motion of a body has been observed to change, either in direction or in velocity, or in both, it has always been possible to trace the change to the influence of other bodies—that is, of force. Furthermore, by diminishing those influences, the changes referred to are correspondingly diminished; and the conclusion has been reached that, could these influences be entirely eliminated, the said changes would not take place at all.*

If, then, a body is at rest, and no unbalanced force acts on it, it will continue at rest; if it is moving in a straight line

with uniform velocity, it will continue to move in the same straight line with the same velocity If a body moves under the action of a force for a certain time, its velocity will constantly change during that time; but, if the force ceases to act, the body will continue to move uniformly *in the direction and with the velocity it had at the instant the force was withdrawn.* It is not necessary, however, in order that the body may continue to move uniformly, to withdraw the force or forces acting on it. the same result will be obtained if other forces are introduced equal and opposed to those already acting. An instance of this is afforded by the motion of a train· at first, the tractive force of the engine is greater than the combined resistance of friction and the air, and the velocity of the train constantly increases, but, as will be explained elsewhere, the resistance increases with the velocity, so that there is a moment when the traction of the engine is balanced by the resistance, and from that moment on, the train moves uniformly.

50. Conversely, *if either the velocity or the direction of motion of a body changes, the body must be under the action of unbalanced forces,* for, according to the law of inertia, rectilinear motion with uniform velocity (in which rest must be included as a special case) is the only possible motion of a body not acted on by unbalanced forces. Thus, if a body moves in a curve, it may be concluded that, whatever its velocity may be, some unbalanced force or forces must be constantly acting on the body. The moment these forces cease to act, the body will continue to move uniformly in the direction of the tangent to the path at the point occupied by the body at that moment. For example, if a stone is tied to a string and swung around, the pull of the string will keep the stone moving in a circle; but the moment the string breaks, the stone will fly off on a tangent to this circle. A train is kept on a curve by the resistance of the rails acting against the flanges of the wheels; but if this resistance is not great enough, the wheels get off the rails, and the train, in leaving the curve, moves straight ahead along the tangent.

51. Inertia of Bodies.—The fact that the motions of all bodies follow the first law of motion is often expressed by saying that all bodies possess the property of **inertia.** In this sense, it is very common in mechanics to refer to what the motion of a body is or would be, "by virtue of the inertia of the body", that is, according to the law of inertia. Suppose, for instance, that a particle is moving along the path AB, Fig. 4 It follows from the first law of motion that the particle must be under the action of unbalanced forces, since the path is not straight. Suppose, also, that, by some means, it is found that the velocity of the particle, when at P, is 6 feet per second. Then, we say that, if the forces were suddenly withdrawn (or balanced), the particle, "by virtue of its inertia," would move uniformly along the tangent PT, describing a space of 6 feet in every second

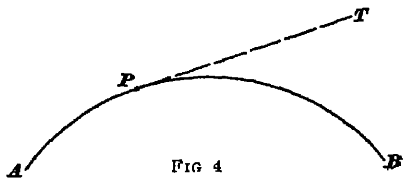


FIG 4

GALILEO'S LAW OF THE INDEPENDENT EFFECT OF FORCES

52. Preliminary Explanation.—Let A, Fig. 5, be a body acted on by a force Suppose the force to be such that, if it acts on the body for a certain time t, and the body is at rest when the force begins to act, the body moves over the path AA' in that time, so that its final position is A'. Suppose, now, that the body, instead of being at rest when the force is applied, is moving along the path AA_1, in such a manner that, if the force did not act, the body would move to A_1 in the time t. It is found from experiment that if, while the body has this motion, the force referred to above constantly acts on it during the time t, the direction of

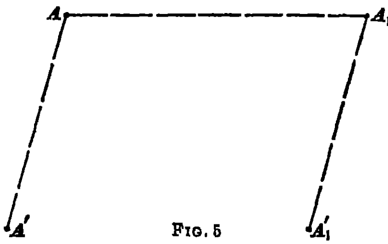


FIG. 5

the force remaining unchanged, the final position of the body after this time is a point A_1', such that the line $A_1 A_1'$ is equal and parallel to $A A'$. In other words, the final position of the body, with respect to the position the body would have occupied if the force had not acted, is the same whether the force starts the body from rest or acts on the body while the latter is in motion and acted on by other forces Had the body been originally at rest, the effect of the force would have been to cause the displacement $A A'$, in the direction of the force, in this case, the final position of the body, had not the force acted, would have been A. If the force had not acted and the body had been moving along $A A_1$, its final position would have been A_1; the effect of the force is to cause the displacement $A_1 A_1'$ of the final position, which displacement is equal and parallel to $A A'$.

53. Statement of Galileo's Law.—From the preceding facts, and from others of a similar character, is derived the following general law, which is **Galileo's law of the independent effects of forces:**

The effect of a force on a body is the same whether the body is at rest or in motion, and, if several forces act simultaneously on a body, each force produces its effect independently of the other forces.

The meaning of this proposition is that each force produces the same amount of displacement, parallel to its direction, as if it acted alone and moved the body from a state of rest

Galileo's law may be otherwise stated as follows:

When a force acts on a body, the position of the body at any time relatively to the position the body would have if the force had not acted, is independent of the motion produced in the body by any other forces.

54. Experimental Verification.—The following experiment affords an easy verification of Galileo's law In Fig. 6, a ball *e* is supported in a cup, the bottom of which is attached to the lever *o* in such a manner that a movement of *o* will swing the bottom horizontally and allow the ball to drop. Another ball *b* rests in a horizontal groove that is provided

with a slit in the bottom. A swinging arm is actuated by the spring *d* in such a manner that, when drawn back, as shown, and then released, it will strike the lever *o* and the ball *b* at the same time This gives *b* an impulse in a horizontal direction and swings *o* so as to allow *e* to fall

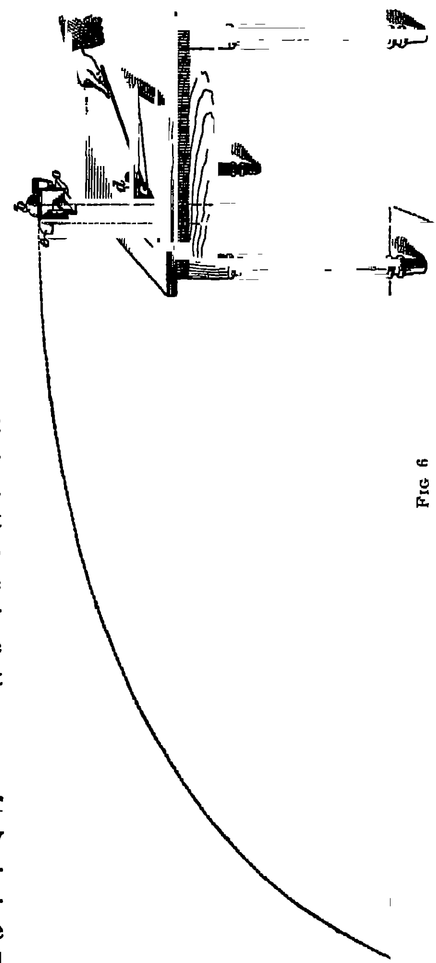


FIG 6

On trying the experiment, it is found that *b* follows a path shown by the curved dotted line, and reaches the floor at the same instant as *e*, which drops vertically This shows that the force that gave the first ball its horizontal movement had no effect on the vertical force that compelled both balls to fall to the floor, the vertical force producing the same effect as if the horizontal force had not acted.

ACCELERATION

55. Change of Velocity Caused by an Unbalanced Uniform Force.—A **uniform force** is a force whose magnitude and direction do not change. Both from the law of inertia and from the definition of force, it follows that the effect of an unbalanced uniform force on a body originally at rest is to set the body in motion and impart to it a continually increasing

velocity Galileo's law affords the means to determine exactly how that increase takes place. Suppose that a uniform force acts on a body originally at rest, imparting to it, in 1 second, a velocity of 10 feet per second This means that, if the force acts during 1 second and then ceases to act, and the body is not interfered with by other forces, the body will, by the law of inertia, continue to move uniformly with a velocity of 10 feet per second If the force, instead of being withdrawn at the end of the first second, acts for 1 second longer, its effect will be independent of the motion already acquired by the body; that is, during the second second, the force will impart to the body a velocity of 10 feet per second, *in addition to the velocity the body had at the end of the first second,* so that, at the end of the second second, the body will have a velocity of 20 feet per second Likewise, the velocity at the end of the third second will be 30 feet per second; and so on.

A similar law of change applies whenever a body is acted on by an unbalanced uniform force, that is, the velocity of the body increases by a fixed amount during every unit of time; or, in other terms, the velocity increases at a constant rate. If, for example, the change of velocity per second is a feet, the change in t seconds will be $a\,t$ feet per second. Of course, it is not necessary to use the second as a unit of time, nor the foot as a unit of length; any other units may be used.

56. Acceleration.—The **acceleration** of a body moving in a straight path under the action of an unbalanced force is the amount by which the velocity of the body in the direction of the force increases in a unit of time.

57. When a body moves under the action of a uniform force, its change of velocity is the same for any two equal intervals of time, as has just been explained. In this case, the acceleration is said to be **uniform**, and the motion is said to be **uniformly accelerated.**

58. In the example given in Art. **55**, the acceleration is 10 feet per second per second The import of this apparently confusing expression is this: If at any moment the

velocity of the body is v feet per second, this means that, if at that moment the force ceased to act, the body would continue to move uniformly with a velocity of v feet per second. If, however, the uniform force continues to act for 1 second longer and then ceases to act, the velocity will have increased to $v + 10$ feet per second; that is, the body, if left to itself, will continue to move uniformly with that velocity, describing $v + 10$ feet every second, or 10 feet more than before. Instead of saying that the acceleration is 10 feet per second per second, it is customary and sufficient to say simply that the acceleration is 10 feet per second, it being understood that the velocity is also measured in feet per second. If the acceleration of a body was said to be 50 miles per hour, the velocity would be understood to be measured in miles per hour, and to be, at the end of any one hour, 50 miles per hour greater than at the beginning of that hour.

59. Formulas for Uniformly Accelerated Motion. Let a body start from rest and move under the action of a uniform force for t seconds. If the acceleration due to this force is a, the velocity at the end of 1 second will be a, at the end of 2 seconds, $2a$; and at the end of t seconds, ta, or at. Denoting by v the velocity at the end of t seconds, we have, then,

$$v = at \qquad (1)$$

$$a = \frac{v}{t} \qquad (2)$$

If, at any instant, the body has a velocity v_0, the velocity v, t seconds after that instant, is given by the formula

$$v = v_0 + at \qquad (3)$$

From this formula follows

$$a = \frac{v - v_0}{t} \qquad (4)$$

The velocity v_0, taken with respect to the interval of time considered, is called the **initial velocity** of the body; and v is called the **final velocity**. It is immaterial how the body has acquired the velocity v_0; in any case, the effect of the applied force is to produce an increase of velocity equal

to $a t$, or $v - v_0$, feet per second, in t seconds The change of velocity produced in 1 second is, therefore, $\frac{v - v_0}{t}$, as expressed by formula **4**, and this agrees with the general definition of acceleration In all this, it is assumed that the force acts in the direction of motion.

If the body starts from rest, $v_0 = 0$, and formulas **3** and **4** become identical with formulas **1** and **2**, respectively.

MASS

60. Force Proportional to Acceleration.—According to Galileo's law, if several forces act simultaneously on a body, each force produces its effect independently of the others Therefore, if the forces act in the same direction, they will produce an acceleration equal to the sum of the accelerations that the forces would produce if each force acted alone. Furthermore, if the forces are equal, the acceleration will be equal to their number multiplied by the acceleration due to any one of them. It is also evident that any force may be supposed to be equivalent to any number of forces acting in the same direction as the given force, and whose sum is equal to the given force Thus, a weight of 5 pounds is equivalent to a weight of 3 pounds and one of 2 pounds, or to a weight of 4 pounds and one of 1 pound, or to five weights of 1 pound each, etc.

Let a force of 3 pounds act on a body, imparting to it an acceleration of 5 feet per second. Then, a force of 12 pounds, which is equivalent to four forces each equal to 3 pounds, will produce an acceleration four times as great, or $4 \times 5 = 20$ feet per second. Any other force will produce an acceleration that will be as many times 5 feet per second as the force contains 3 pounds. For instance, a force of 49 pounds will produce an acceleration of $\frac{49}{3} \times 5 = 81.67$ feet per second, nearly. In general, let a force F_1 impart the acceleration a_1 to a certain body. Let F_n be another force imparting the acceleration a_n to the same body. If F_n is

twice F_1, then will a_n be twice a_1; and, generally, if $F_n = nF_1$ where n is any number, fractional or integral, then $a_n = na_1$. Now,

$$\frac{F_n}{F_1} = \frac{nF_1}{F_1} = n; \ \frac{a_n}{a_1} = \frac{na_1}{a_1} = n;$$

that is,

$$\frac{F_n}{F_1} = \frac{a_n}{a_1}$$

Therefore, *for the same body, any two forces are to each other as the accelerations they impart to the body.*

From the last equation follows

$$\frac{F_n}{a_n} = \frac{F_1}{a_1}$$

61. If several forces F_1, F_2, F_3, etc are capable of producing, respectively, the accelerations a_1, a_2, a_3, etc. in the same body, then,

$$\frac{F_1}{a_1} = \frac{F_2}{a_2} = \frac{F_3}{a_3}, \text{ etc.}$$

Each of these quotients may be considered to represent the force necessary to give the body an acceleration of 1 foot per second (or, in general, 1 unit of length per unit of time). If the value of this force, or the common value of the preceding quotients, is denoted by m, then

$$m = \frac{F_1}{a_1} = \frac{F_2}{a_2} = \frac{F_3}{a_3}, \text{ etc.;}$$

and hence,

$$F_1 = ma_1, F_2 = ma_2, F_3 = ma_3, \text{ etc.}$$

In general, if F is any force producing in a body the acceleration a, we have

$$m = \frac{F}{a} \qquad (1)$$

and

$$F = ma \qquad (2)$$

62. Definition of Mass.—Experience teaches that it requires different forces to produce the same acceleration in different bodies. Therefore, the value of m is different for different bodies. When two equal forces produce the same acceleration in two bodies, the two bodies are said to have the same *mass*. If one body requires a greater force than

another in order to receive a certain acceleration, the former body is said to have a greater mass than the latter.

Mass, then, is that property of bodies on which alone the acceleration they receive when under the action of given forces depends. The acceleration, of course, depends on the applied force also, what the definition means is that, so long as the force remains the same, the acceleration varies with only that property of bodies called *mass*

According to this definiton, the mass of a body may be measured by the force necessary to impart to the body an acceleration of a unit of length per unit of time For this reason, the factor m of formulas **1** and **2** of Art. **61** is called a *measure of the mass of the body*, or, for shortness, *the mass of the body*. It should be remembered that m has different values for different bodies.

63. Determination of the Mass of a Body—Acceleration Due to Gravity.—The mass of a body has to be determined experimentally. A known force is applied to the body for a certain time, and the velocity at the end of the time is ascertained by some method Then the acceleration is found by dividing the velocity by the time (formula **2** of Art. **59**), and the mass by dividing the applied force by the acceleration (formula **1** of Art. **61**). The mass is, of course, expressed in the same units as the force, and its numerical value further depends on the unit of time and on the unit of length, these units being involved in acceleration. In this work, mass will, unless otherwise stated, be referred to the pound, the foot, and the second as units.

The most convenient force to use for determining the mass of a body is the force of gravity—that is, the weight of the body. It has been ascertained by actual experiment that a body falling freely *in vacuo* follows the laws of uniformly accelerated motion. This might have been anticipated; for, while the body falls, it is under the action of a uniform force (gravity), which is measured by the weight of the body.

The velocity of a falling body has been found to increase at the constant rate of about 32.16 feet per second per second;

in other words, when acting on a falling body, gravity produces an acceleration of about 32.16 feet per second. As this acceleration is due to the weight of the body, we have, denoting this weight by W (from formula **1** of Art. **61**),

$$m = \frac{W}{32.16}$$

64. The acceleration produced in a body by the force of gravity is called the **acceleration due to gravity**, or the **acceleration of gravity.** The magnitude of this acceleration decreases from the pole, where it is about 32.26 feet per second, toward the equator, where it is about 32.09 feet per second For the United States, its average value is 32.16 feet per second. This value will be used here, unless a different value is especially stated.

65. The acceleration due to gravity is usually denoted by g. If the weight of a body is W, we have,

$$m = \frac{W}{g} \qquad (1)$$

and

$$W = mg \qquad (2)$$

The weight varies proportionately to g, so that the value of m, as determined from formula **1**, is always the same, as it should be. It is, of course, understood that the weight W is the weight of the body at the place where the acceleration of gravity is g.*

Substituting in formula **2** of Art. **61** the value just found for m,

$$F = \frac{Wa}{g} \qquad (3)$$

which gives the force necessary to produce a given acceleration a in a body having a given weight W.

Both in formula **3** of this article and in formula **2** of Art. **61**, the force F is the force acting in the direction of

*The exact value of g, at any particular place, is determined by careful observations with a pendulum, to which formulas derived in advanced mechanics are applied. The following formula gives a fairly approximate value of g (feet per second) at any point whose latitude is L and whose elevation above sea level is h (feet):

$$g = 32.173 - 084 \cos L - .000002\ h$$

motion. If there is another force acting in an opposite direction, F must be understood to be the difference between the two. How forces acting on the same body or particle are combined will be explained further on.

EXAMPLE 1 —A block of any material is placed on a horizontal smooth surface (that is, a surface supposed not to offer any frictional resistance, or whose frictional resistance may be neglected), and is pulled by a string for 4 seconds, a registering apparatus (called a *dynamometer*) placed between two portions of the string shows that the pull is 3 pounds The weight of the block being 20 pounds, it is required to find (*a*) the mass of the block, (*b*) the acceleration of its motion; (*c*) its velocity at the end of the fourth second

SOLUTION —(*a*) The mass of the block is found by formula **1**. Taking g equal to 32 16,

$$m = \frac{20}{32\ 16} = 622 \quad \text{Ans.}$$

(*b*) By formula **2** of Art **61**,

$$a = \frac{F}{m} = \frac{3}{.622} = 4\ 823 \text{ ft per sec.} \quad \text{Ans.}$$

(*c*) By formula **1** of Art **59**,

$$v = 4\ 823 \times 4 = 19\ 292 \text{ ft per sec} \quad \text{Ans}$$

EXAMPLE 2 —A uniform force acts on a body originally at rest for 10 seconds, at the end of which the velocity of the body is 40 feet per second, the force then ceases to act, and a force of 15 pounds is applied for 5 seconds, at the end of which the velocity of the body is 75 feet per second To find (*a*) the weight of the body, (*b*) the force that acted during the first 10 seconds

SOLUTION —(*a*) During the second interval (5 seconds), the velocity has changed from 40 to 75 ft per sec Therefore, the acceleration, by formula **4** of Art **59**, is $\frac{75 - 40}{5} = 7$ ft. per sec. ($= a$). The mass of the body is $\frac{F}{a} = \frac{15}{7}$, and the weight is

$$mg = \frac{15 \times 32.16}{7} = 68.914 \text{ lb.} \quad \text{Ans.}$$

(*b*) The acceleration due to the first force is (formula **2** of Art. **59**), $40 \div 10 = 4$ ft per sec By formula **2** of Art. **61**, the force necessary to produce this acceleration is

$$m \times 4 = \frac{15}{7} \times 4 = 8.5714 \text{ lb} \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE

1. Find the force necessary to start a weight of 15 pounds from rest and give it a velocity of 80 feet per second in 4 seconds
Ans $F = 9\ 3284$ lb.

2 An engine and train of cars having an aggregate weight of 150 tons start from rest and move for 1 minute (60 seconds) over a level track Assuming the traction of the engine to exceed the resistances by 2 50 tons, find the velocity of the train at the end of the minute
Ans $v = 32\ 16$ ft per sec

3 A body has an initial velocity of 20 feet per second. A force of 40 pounds is applied, and after 12 seconds the velocity is found to be 85 feet per second Find (*a*) the acceleration due to the force, (*b*) the weight of the body
Ans $\begin{cases} (a) & a = 5\ 4167 \text{ ft per sec.} \\ (b) & W = 237.49 \text{ lb} \end{cases}$

4 A body weighing 980 pounds moves for a certain time under the action of a uniform force of 50 pounds. At the end of that time, another force of 50 pounds is applied in the opposite direction After the application of the second force (the first not being removed), the body moves over 300 feet in 15 seconds Find the time during which the first force acted alone
Ans $t = \frac{v}{a} = \frac{300 - 15}{\frac{F}{W}g} = 12.189$ sec.

LAW OF ACTION AND REACTION

66. Action and Reaction.—Force has been defined as the action of a body on another, producing, or tending to produce, a change in the motion of the latter body. When the weight of a body is spoken of as the action of the earth on the body, we are concerned only with the effects of this action on the body in question, and for this reason disregard whatever action the body may, in its turn, exert on the earth. In the same manner, we say that a magnet attracts a piece of iron; this is sufficient for us to know, if the only thing being dealt with is the condition and motion of the piece of iron. But it must not be concluded from these common and convenient forms of expression that a body can act on another without itself being acted on by the other body. All action between bodies is mutual. Take, for instance, a magnet, hold it in the hand, and bring it near a small piece of iron· the iron will immediately move toward the magnet and

adhere to it. Now, place the magnet on the table and hold the piece of iron in the hand, bringing it near the magnet: the magnet will move toward the piece of iron and adhere to it. If both the magnet and the piece of iron are placed on corks and let float on the surface of water, they will both move, each toward the other, and adhere to each other. Furthermore, in this motion, the acceleration of the magnet and that of the iron are found to be in the inverse ratio of the masses of the two bodies. This shows that the two bodies are acted on by the same force (Art. 62), or that the action of the magnet on the iron is equal to the action of the iron on the magnet.

When a body is at rest, it is evident that the forces acting on it must balance among themselves (law of inertia) Take a body weighing 4 pounds lying on a flat surface If the weight of the body were not balanced, the body would move; the surface, therefore, must exert an upward force equal to 4 pounds, in order to keep the body from falling If we press against the wall with a force of 10 pounds, the wall presses on the hand with the same force, but in an opposite direction. A weight of 20 pounds hanging from a string will evidently pull the string downwards with a force of 20 pounds; but the string must pull the weight upwards with the same force, as otherwise the weight would fall.

In general, whenever one body acts on another, the latter body acts on the former with the same force, but in the opposite direction.

In considering the mutual action of two bodies, we are usually concerned with the condition of only one of them; the force acting on it is called the **action**, while the force exerted by it on the other body is called the **reaction**.

67. Statements of the Law of Action and Reaction. From the facts stated in the preceding article, the following general law has been derived; it is known as **the law of action and reaction**, and was first stated by Newton:

To every action, there is always an equal and opposite reaction.

68. As the accelerations produced in two bodies by the same force are inversely proportional to the masses of the two bodies, the law of action and reaction may be stated in the following terms

Whenever two bodies act on each other, they produce, or tend to produce, in each other accelerations in opposite directions, and these accelerations are inversely proportional to the masses of the two bodies.

This happens whether the two bodies act on each other by means of a string stretched between the two, or by means of a rod connecting them, or through some unknown medium, as in the case of a magnet and a piece of iron. In every case where a body moves another, the latter moves, or tends to move, the former

69. Stress.—When only the action of one body on another is considered, this action is called force. If the mutual action of two bodies, or two parts of a body, is

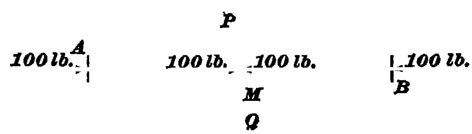


FIG 7

considered, that action is called **stress**. A stress includes a pair of equal and opposite forces, an action and a reaction, a force and a counterforce. A stress is measured by the magnitude of either of the forces of which it consists.

In applied mechanics, the term stress is generally restricted to the forces acting between two parts of a body Thus, when a string is pulled at both ends with a force of 10 pounds, every portion of the string is under a stress of 10 pounds; for, if the two portions of the string are considered as being situated on the two sides of a plane perpendicular to its direction, each portion will be pulled away from the other with a force of 10 pounds. Therefore, to keep the two portions from separating, each must pull the other with a force of 10 pounds. Again, consider the bar AB, Fig. 7, pushed from each end with a force of 100 pounds. Imagine the bar

to be divided by a plane PQ when the portion AM presses on BM with a force of 100 pounds toward the right, the portion BM must press on AM with a force of 100 pounds toward the left.

In these examples, the string is under a stress of 10 pounds, and the bar under a stress of 100 pounds.

70. Tension, Compression, Pressure.—When, as in the case of the string just considered, the forces tend to pull each portion of a body from the other, or to lengthen the body on which they act, the stress is called **tension.** When, as in the case of the bar, the forces tend to move each portion toward the other, or to shorten the body on which they act, the stress is called **pressure** or **compression.** The term pressure, however, is more commonly applied to the stress between two contiguous bodies, and the term compression to the stress between two contiguous parts of the same body. In the example of the bar, the stress is ordinarily called compression. In the case of a heavy body resting on a table, the force exerted by the body on the table and by the table on the body is a pressure.

EXAMPLE 1.—Two bodies M_1 and M_2, Fig. 8, of masses m_1 and m_2, respectively, rest on a smooth horizontal surface, and are connected by a string S_2. A force of F pounds is applied to M_1 by means of a string S_1 in the direction shown. To find the acceleration of the resulting motion, and also the tension in each string.

FIG. 8

SOLUTION.—In all problems of this kind, the strings are supposed to be rigid and of no mass or weight; if the mass and stretching of each string were considered, the problem would be more complicated. In almost all cases that occur in practice, the mass of the string, rope, chain, etc. may be neglected without any sensible error; and the same is true of the extensibility, or stretching capacity, of the connection. For example, the mass of the coupler between two railroad cars need not be considered in determining the pull between one car and the other; nor is it necessary to take into account the amount of stretching in the coupler.

Since the two bodies M_1 and M_2 are rigidly connected, they must move with the same acceleration. Let this acceleration be a, and let

the tensions in the strings S_1 and S_2 be T_1 and T_2, respectively Since the body M_1 is pulled by the string S_1 with a force equal to F, it must pull the string toward the left with the same force, so that T_1 is evidently equal to F Also, M_1 acts on M_2 through the string S_2 and imparts to it an acceleration a toward the right, therefore, the force exerted by M_1 on M_2 is (from formula **2** of Art **61**) $m_2 a$ By the law of action and reaction, M_2 exerts on M_1 the same force, $m_2 a$, but directed toward the left The tension in S_2 is, therefore, $T_2 = m_2 a$ The net force acting on M_1 is the difference between F and $m_2 a$, or $F - m_2 a$. Consequently, since the resulting acceleration is a, we have, by formula **2** of Art **61**, $F - m_2 a = m_1 a$; whence,

$$a = \frac{F}{m_1 + m_2} \qquad (1)$$

Also,
$$T_2 = m_2 a = \frac{m_2 F}{m_1 + m_2} \qquad (2)$$

The same results may be obtained by assuming at the outset that the two masses m_1 and m_2 are equivalent to one single mass $(m_1 + m_2)$ acted on by the force F From this, equation (1) follows at once as a special case of formula **2** of Art **61**. Knowing the acceleration of M_2, which is a, the unbalanced force T_2 acting on M_2 is found by formula **2** of Art **61**:

$$T_2 = m_2 a = \frac{m_2 F}{m_1 + m_2}$$

which is the same as before.

EXAMPLE 2 —A body M_1, Fig 9, whose weight is W_1 pounds, is attached to a string, the latter is passed over a smooth peg P, and then tied to a second body M_2 having a weight of W_2 pounds and free to move along a smooth horizontal surface To find the pull in the string and the resulting acceleration of the two bodies.

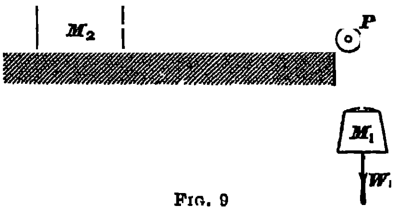


FIG. 9

SOLUTION —This problem does not differ from the preceding in principle, and the method of solving it is identically the same, but it has been given in order to caution against a common mistake made by beginners The object of the peg is simply to change the direction of the force, it has no effect on the magnitude of the pull, which is the same in the vertical as in the horizontal portion of the string It is to be carefully borne in mind, however, that the pull on the string is not the weight of M_1, and that therefore the force pulling M_2 is not W_1 pounds, as might appear at first sight. The weight W_1 is a force acting on the bodies M_1 and M_2, and, like the force F in the preceding example, has to move these two bodies. If there were

no body M_2, the unbalanced force acting on M_1 would be W_1, but, in the present case, this force is diminished by the reaction of M_2 transmitted through the string to M_1. If the pull in the string is denoted by T and the common acceleration of the two bodies is denoted by a, the unbalanced force acting on M_1 is, as before, $W_1 - T$, and the unbalanced force acting on M_2 is simply T. The mass of M_1 is $\frac{W_1}{g}$ (formula 1 of Art. 65), and the mass of M_2 is $\frac{W_2}{g}$. Substituting in the formulas of the preceding solution, we obtain

$$a = \frac{W_1}{\frac{W_1}{g} + \frac{W_2}{g}} = \frac{W_1 g}{W_1 + W_2} \qquad (1)$$

$$T = \frac{W_1 \frac{W_2}{g}}{\frac{W_1}{g} + \frac{W_2}{g}} = \frac{W_1 W_2}{W_1 + W_2} \qquad (2)$$

EXAMPLE 3.—A body weighing W pounds lies on the surface of a horizontal table. Both the table and the body are moving upwards with an acceleration of a feet per second. To find the pressure of the body on the table.

SOLUTION.—The pressure of the body on the table is equal to the pressure of the table on the body, which is the total upward force acting on the body; call this force P. The downward force acting on the body is W. Therefore, the unbalanced force producing the acceleration a is $P - W$, and, as $P - W = am = \frac{Wa}{g}$, we find, for the downward pressure exerted by the body,

$$P = \frac{Wa}{g} + W = \frac{W(g + a)}{g} \qquad (1)$$

This shows that the weight of the body is increased by the amount $\frac{Wa}{g}$. If the table is moving downwards, a is negative, and $P = \frac{W(g - a)}{g}$. If in this case $a = g$, then, $P = 0$, or there is no pressure on the table. If a is negative (downward) and greater than g, say $a = g + a'$, then,

$$P = \frac{W[g - (g + a')]}{g} = -\frac{Wa'}{g} \qquad (2)$$

This shows that, if the body remains in contact with the surface of the table, it will exert an upward pressure equal to $\frac{Wa'}{g}$; but this can only be done by placing the body under the table. In this case, it is evident that, since the table tends to move faster than the body, the latter must offer a resistance, or exert an upward pressure on the table.

This problem should be carefully studied, as it is a good illustration of how the results of mathematical formulas are to be interpreted.

EXAMPLES FOR PRACTICE

1. In example 1 of Art **70**, the force F is 500 pounds, and M_1 and M_2 weigh 8 tons and 3 tons, respectively Find (*a*) the resulting acceleration, in feet per second, (*b*) the tension on the rope, in pounds (1 ton = 2,000 pounds).

Ans $\begin{cases} (a) & a = 731 \text{ ft per sec} \\ (b) & T = 136\ 36 \text{ lb.} \end{cases}$

2 With the weights of M_1 and M_2 as in example 1, find: (*a*) the force (tons) necessary to give the two bodies an acceleration of 10 feet per second, (*b*) the tension in the rope, in tons

Ans $\begin{cases} (a) & F = 3\ 4204 \text{ tons} \\ (b) & T = .9328 \text{ ton} \end{cases}$

3. In Fig. 10, the three bodies M_1, M_2, M_3, weighing 18 tons, 16 tons, and 20 tons, respectively, are connected by the ropes S_2

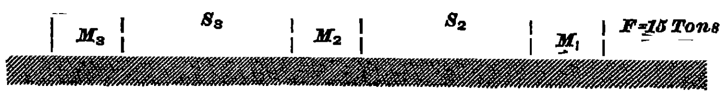


FIG 10

and S_3 A force F of 15 tons is applied to M_1 as shown. Find. (*a*) the resulting acceleration a, in feet per second, (*b*) the tension T_3 in the rope S_3, (*c*) the tension T_2 in the rope S_2

Ans $\begin{cases} (a) & a = 8\ 9333 \text{ ft per sec.} \\ (b) & T_3 = 5\ 5556 \text{ tons} \\ (c) & T_2 = 10\ 000 \text{ tons} \end{cases}$

4 A body that on the surface of the earth weighs 10 pounds, is weighed in an ascending balloon by means of a spring balance, and found to weigh 15 pounds With what acceleration is the balloon ascending? [See equation (1) in example 3 of Art. **70.**]

Ans $a = 16\ 08$ ft per sec.

5. An engine pulls a train of seven cars weighing 30 tons each The total force of the locomotive—that is, the total force acting on the train (the engine not included)—is 8 tons. Assuming a constant resistance of 10 pounds per ton of train (engine and tender excluded) during the first minute, find (*a*) the acceleration a of the train, in feet per second, (*b*) the velocity v of the train at the end of the first minute, in miles per hour, (*c*) the tension T in the coupler connecting the second car to the first.

Ans. $\begin{cases} (a) & a = 1\ 0643 \text{ ft per sec} \\ (b) & v = 43.541 \text{ mi per hr.} \\ (c) & T = 6.8571 \text{ tons} \end{cases}$

IMMEDIATE CONSEQUENCES OF THE PRECEDING LAWS

IMPORTANT FORMULAS

71. Space Passed Over in Uniformly Accelerated Motion.—Let a body start from rest and move during t seconds under the constant action of a force acting in the direction of motion. Let a be the acceleration of the body, and v the velocity at the end of the time t. As explained in Art. **59**, v is equal to $a\,t$. It can be shown by the use of advanced mathematics that the space s described by the body in the time t is the same as if the body had moved during that time with a constant velocity equal to one-half the actual final velocity, or $\frac{v}{2}$; that is,

$$s = \frac{v}{2}\,t \qquad (1)$$

If, in the second member of this formula, v is replaced by its equivalent $a\,t$, the result is $\frac{a\,t}{2}\,t = \frac{1}{2}\,a\,t^2$. Therefore,

$$s = \tfrac{1}{2}\,a\,t^2 \qquad (2)$$

This is one of the most important formulas in mechanics and should be memorized.

72. If, in formula **1** of Art. **71**, $t = 1$, then, $s = \frac{1}{2}\,a$; or, *the space described during the first second is numerically equal to one-half the acceleration.*

73. From the general formula $v = a\,t$, we get $t = \frac{v}{a}$. This value substituted in formula **1** of Art. **71** gives $s = \frac{1}{2}\,\frac{v^2}{a}$, whence, $v^2 = 2\,a\,s$, and

$$v = \sqrt{2\,a\,s}$$

which gives the final velocity at the end of a given space.

74. To find the space and the velocity in terms of the moving force and the mass or the weight of the moving body, we

have, by formula 2 of Art. 61, and formula 1 of Art. 65, $a = \frac{F}{m} = \frac{F}{W} g$. These values in formula 2 of Art. 71 give

$$s = \frac{1}{2} \frac{F}{m} t^2 = \frac{1}{2} \times \frac{F}{W} g t^2 \qquad (1)$$

The same values in the formula of Art. 73 give

$$v = \sqrt{\frac{2 F s}{m}} = \sqrt{\frac{2 F g s}{W}} \qquad (2)$$

75. Modification of the Formulas When the Body Has an Initial Velocity.—If, before the force begins to act, the body has an initial velocity v_0, or, what is the same thing, if only an interval of time during which the velocity changes from v_0 to v is considered, the preceding formulas must be modified as follows:

As the force has no effect on the motion already acquired by the body, the latter, having a velocity v_0, will describe, by virtue of its inertia alone, during the time t, a space equal to $v_0 t$; to this must be added the space described by the body in virtue of the action of the force, as given by formula 2 of Art. 71. Therefore, the total space is given by the formula

$$s = v_0 t + \tfrac{1}{2} a t^2 \qquad (1)$$

or, putting $a = \frac{v - v_0}{t}$ (formula 4 of Art. 59),

$$s = v_0 t + \tfrac{1}{2}(v - v_0) t = \tfrac{1}{2}(v + v_0) t \qquad (2)$$

which is the mean of the initial and the final velocity multiplied by the time.

76. The equation $a = \frac{v - v_0}{t}$ gives $t = \frac{v - v_0}{a}$. Substituting this value in formula 2 of Art. 75,

$$s = \tfrac{1}{2}(v + v_0) \times \left(\frac{v - v_0}{a}\right) = \frac{v^2 - v_0^2}{2 a}$$

whence, $v^2 - v_0^2 = 2 a s$, and

$$v = \sqrt{v_0^2 + 2 a s}$$

This gives the final velocity in terms of the initial velocity, the space, and the acceleration.

To find s and v in terms of force and mass, it is only necessary to substitute $\frac{F}{m}$ for a.

77. Falling Bodies.—It has been explained (Art. **63**) that, when a body falls freely under the action of gravity, its acceleration is g feet per second, the value of g being nearly constant and equal to 32 16. The preceding formulas apply to falling bodies, for a falling body is simply a body moving under the action of a force equal to its own weight, with a uniform acceleration equal to g. It is, therefore, not strictly necessary to write new formulas for the motion of falling bodies. As, however, it is customary to use the symbol g for the acceleration of a falling body, and h for the space described, or height fallen through, it is convenient to have the general formulas expressed in terms of these symbols. By putting h for s and g for a in formula **1** of Art. **59**, in formulas **2** and **1** of Art. **71**, and in the formula given in Art. **73**, the following formulas are obtained for a body falling freely from rest during t seconds:

$$v = gt \qquad (1)$$

$$h = \tfrac{1}{2}gt^2 \qquad (2)$$

$$h = \tfrac{1}{2}vt \qquad (3)$$

$$v = \sqrt{2gh} \qquad (4)$$

For a body falling for t seconds, after having acquired a velocity v_0, formula **3** of Art. **59**, formulas **1** and **2** of Art. **75**, and the formula of Art. **76**, give

$$v = v_0 + gt \qquad (5)$$

$$h = v_0 t + \tfrac{1}{2}gt^2 = (v_0 + \tfrac{1}{2}gt)t \qquad (6)$$

$$h = \tfrac{1}{2}(v_0 + v)t \qquad (7)$$

$$v = \sqrt{v_0^2 + 2gh} \qquad (8)$$

78. It should be remembered that, theoretically, these formulas are exactly correct only for bodies falling *in vacuo*. When bodies fall through the air, the latter offers a resistance that varies with the surface of the body; hence, all bodies do not fall in air with exactly the same acceleration g; but in a vacuum they do. This has been verified by the following experiment: A feather and a ball of lead are supported at the upper (inside) part of a long tube from which the air has been removed; by a convenient arrangement the support is quickly turned from the outside, so that both

bodies will begin to fall at exactly the same time; and it is found that they reach the bottom of the tube simultaneously; nor does the ball get ahead of the feather during any portion of the time of falling. Again, if a paper disk is placed on top of a dollar, and the latter let fall, the paper will remain constantly on the dollar and fall in the same time. The reason is that in this case the paper does not encounter any more resistance than does the coin.

EXAMPLE 1—What is the velocity at the end of 30 seconds of a body moving with an acceleration of 8 feet per second?

SOLUTION—Substituting the given values in formula **1** of Art **59**,

$$v = 8 \times 30 = 240 \text{ ft per sec.} \quad \text{Ans}$$

EXAMPLE 2—A force of 10 pounds starts a body from rest and causes it to move over 65 feet If the weight of the body is 40 pounds, what is the final velocity?

SOLUTION.—By formula **3** of Art **65**,

$$a = \frac{Fg}{W} = \frac{10 \times 32\ 16}{40} = 8\ 04 \text{ ft. per sec.}$$

Substituting the known values in the formula of Art **73**,

$$v = \sqrt{2as} = \sqrt{2 \times 8\ 04 \times 65} = 32\ 33 \text{ ft per sec} \quad \text{Ans.}$$

EXAMPLE 3.—A ball is thrown downwards with a velocity of 5 feet per second from a tower 200 feet high. What will be its velocity when it reaches the ground?

SOLUTION—Here, the initial velocity $v_0 = 5$ ft per sec, and the height $h = 200$ ft., are given. Substituting these values in formula **8** of Art **77**,

$$v = \sqrt{5^2 + 2 \times 32\ 16 \times 200} = 113\ 53 \text{ ft per sec.} \quad \text{Ans}$$

EXAMPLE 4—If a ball dropped from a bridge reaches the water in 2 seconds, at what height is the bridge above the surface of the water?

SOLUTION—From formula **2** of Art. **77**, the height through which the ball drops is found to be

$$\tfrac{1}{2} g t^2 = \tfrac{1}{2} \times 32.16 \times 2^2 = 64\ 32 \text{ ft} \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE

1. A body starts from a state of rest and moves with an acceleration of 20 feet per second for 30 seconds. Over what distance has the body passed? Ans. 9,000 ft.

2. A body weighing 160.8 pounds is started from rest by a force of 40 pounds. Over what space has the body passed if the final velocity is 16 feet per second? Ans. 16 ft.

3 A body having an initial velocity of 25 feet per second moves, with an acceleration of 8 feet per second, over a distance of 29 feet Find the final velocity of the body. Ans 33 ft per sec

4 A body falls freely for 2 3 seconds If the initial velocity of the body is 5 feet per second and its final velocity is 79 feet per second, through what distance has the body fallen? Ans 96 6 ft

5 A body having an initial velocity of 10 feet per second falls freely for 12 seconds Through what distance does the body pass? Ans 2,435 5 ft

RETARDATION

79. Uniformly Retarded Motion.—When a body is moving in a certain direction and a force acts in the opposite direction, the effect of the force is to diminish the velocity by equal amounts in equal times, and the motion is said to be **uniformly retarded.** The amount by which the velocity is decreased per unit of time is called the **retardation** due to the force.

According to Galileo's law, if a force acting on a body at rest produces in it an acceleration a, the same force acting on the same body, when the latter is in motion, will produce the same acceleration a and the velocity $v = a\,t$, in its own direction, whatever the previous direction of the motion may have been Therefore, if the force acts in a direction opposite to that in which the body is moving, and if the latter has an initial velocity v_0, the velocity after t seconds will be $v_0 - a\,t$. Retardation, then, is nothing but *negative acceleration*, and all formulas relating to uniformly accelerated motion apply to uniformly retarded motion, by simply changing the sign of a. For this reason, the term acceleration, when taken in its most general sense, includes retardation as a special case, and the latter term is seldom used in mechanics.

80. Law of Retardation.—Let a body be moving in a straight line, and let its velocity at a certain moment be v_0. If at that moment a force is applied in the opposite direction, and allowed to act during t seconds, the final velocity v

and the space s described are given by the following formulas (See Arts 59 and 75):

$$v = v_0 - a\,t \qquad (1)$$
$$s = v_0 t - \tfrac{1}{2} a\,t^2 \qquad (2)$$

To find the time in which the body will be brought to rest, v must be equal to zero; that is, $v_0 - a\,t = 0$, or $t = \frac{v_0}{a}$. Substituting this value in equation (2), and denoting by s' the space described during the time t, we have,

$$s' = \frac{v_0^2}{a} - \tfrac{1}{2} a \frac{v_0^2}{a^2} = \frac{v_0^2}{2a}$$

If the velocity v_0 is supposed to have been generated by a force equal to the retarding force, the time required for the force to produce this velocity must evidently have been $\frac{v_0}{a}$, which is the same as the time required for the retarding force to destroy the velocity v_0. The space described during the action of the force, up to the time the velocity v_0 is attained, is given by the formula

$$s = \tfrac{1}{2} a\,t^2 = \tfrac{1}{2} a \frac{v_0^2}{a^2} = \frac{v_0^2}{2a} = s'$$

It follows, therefore, that, *if a uniform force acts on a body during a certain time and carries it through a certain distance, an equal but opposite force, if applied after the first force, will destroy the velocity generated by the first force in the same time and in the same space (that is, after the body has described the same space) required by the first force to generate the velocity in question.*

SOLUTION OF PROBLEMS

81. Fundamental Equations of Motion.—The following are the three fundamental equations of motion:

$$F = m\,a \qquad (1)$$

$$\left.\begin{aligned} v - v_0 &= a\,t \\ s &= v_0 t + \tfrac{1}{2} a\,t^2 \end{aligned}\right\} \qquad (2)$$

They form two groups, (1) and (2), one consisting of one equation and the other of two. In group (1), there are three quantities; and as there is only one equation, two of the

quantities must be known before the other can be determined. If F is to be determined, m and a must be known, if m is to be determined, F and a must be known, etc. In group (2), there are five quantities, and, as there are only two equations, three of the quantities must be known in order to determine the other two; the process of finding these two is simply the process of solving two equations with two unknown quantities.

If the two groups are taken together, it will be observed that a is common to both If two of the quantities in group (1) are given, a becomes known, and to solve group (2), only two more quantities of that group have to be given Or, if three of the quantities of group (2) are given, a becomes known, and only one of the quantities of group (1) (either F or m) is required to find the other. The process of elimination is exceedingly simple. When there is no initial velocity, $v_0 = 0$.

82. General Solution of Some Important Problems. As an illustration of the use of the fundamental equations, let it be required to find F, a, and t, when m, v, v_0, and s are given.

In group (2), the quantities v, v_0, and s are known. The unknown quantities are a and t. From the first equation of the group is obtained $a = \frac{v - v_0}{t}$; and from the second, $a = \frac{2(s - v_0 t)}{t^2}$. Equating these two values of a, transposing, and canceling the common factor t,

$$(v - v_0)\, t = 2\,(s - v_0 t);$$

whence
$$t = \frac{2s}{v + v_0}$$

which is the same value that would have been obtained from formula 2 of Art 75. This value of t substituted in the equation $a = \frac{v - v_0}{t}$ gives

$$a = \frac{v - v_0}{\frac{2s}{v + v_0}} = \frac{(v - v_0)(v + v_0)}{2s} = \frac{v^2 - v_0^2}{2s}$$

Finally,
$$F = ma = \frac{m(v^2 - v_0^2)}{2s}$$

83. Again, let m, v, v_0, and t be given, and let it be required to find F, s, and a.

The value of a follows at once from the relation $v - v_0 = a\,t$, which gives $a = \frac{v - v_0}{t}$. Substituting this value in the second equation of group (2), formula **2** of Art. **75** is again found. From group (1) is obtained

$$F = m\,a = \frac{m(v - v_0)}{t}$$

The values for F given in this and the preceding article are very important, and should be memorized.

Example 1 —A body weighing 250 pounds is moving with a velocity of 25 feet per second, a force acting in the same direction through a space of 75 feet changes the velocity to 50 feet per second To find the magnitude of the force

Solution —Here, $m = \frac{250}{32\ 16}$ (formula **1** of Art **65**), $v_0 = 25$, $v = 50$, and $s = 75$ Substituting these values in the formula given in Art **82**,

$$F = m\,a = \frac{m(v^2 - v_0^2)}{2s} = \frac{\frac{250}{32\ 16}(50^2 - 25^2)}{2 \times 75} = \frac{250 \times (2{,}500 - 625)}{32\ 16 \times 2 \times 75}$$
$$= 97.17 \text{ lb. Ans}$$

Example 2 —A body whose weight is 192 96 pounds moves with a velocity of 15 feet per second. To find the magnitude of a force necessary to change the velocity to 45 feet per second, that force acting during 15 seconds in the direction of motion

Solution —Here, $m = \frac{192.96}{32\ 16} = 6$, $v_0 = 15$, $v = 45$, and $t = 15$. Substituting these values in the formula given in this article,

$$F = m\,a = \frac{m(v - v_0)}{t} = \frac{6(45 - 15)}{15} = 12 \text{ lb. Ans.}$$

EXAMPLES FOR PRACTICE

1 A body weighing 100 pounds moves for 6 seconds along a smooth horizontal surface under the action of a force of 10 pounds Find the space passed over. Ans. $s = 57.888$ ft.

2 A body moving with uniformly accelerated motion passes over 100 feet in 12 seconds. What is the acceleration?

Ans. $a = 1.3889$ ft. per sec.

3. What force is necessary to give a body weighing 6 tons a velocity of 10 feet per second in 75 feet?

Ans. $F = \frac{W v^2}{2 g s} = .12438$ T. $= 248.76$ lb.

4. A body starts from rest and moves with an acceleration of 40 feet per second. Find the velocity after the body has passed over 75 feet. Ans. $v = 77.46$ ft. per sec.

5. Find the time in which the body in the preceding example has described the space of 75 feet. Ans. $t = 1.9365$ sec.

6. A body is thrown vertically downwards from a height of 10,000 feet and reaches the ground in 6 seconds. Find the velocity with which the body was projected. (Use formula **6** of Art. **77**.)

Ans. $v_0 = 1{,}570.2$ ft. per sec.

COMPOSITION AND RESOLUTION OF FORCES

84. Point of Application, Line of Action, and Direction of a Force.—When a force is applied to a particle or material point, the particle or point is called the **point of application of the force.**

The **line of action** of a force is the straight line along which the force tends to move its point of application. The **direction** of the force is the direction of this line (see Art. **37**).

85. Graphic Representation of Forces.—As a great many mechanical problems are solved by means of geometry,

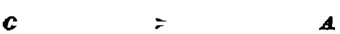


FIG. 11

it is convenient, and almost necessary, to represent forces **graphically**—that is, by means of lines. A force is represented graphically by drawing a line parallel to the line of action of the force, and of a length proportional to the magnitude of the force, the direction of the latter being indicated by an arrowhead marked on the line. In Fig. 11, O is the point of application of a force, and the force is represented by OA, whose length is as many units of length as there are pounds in the force. The arrow indicates that the force tends to move O from O toward A along the line OA. If the force is 100 pounds, OA may be made $\frac{100}{8}$ inch, or $\frac{100}{16}$ inch, or 100 millimeters, etc.; the unit of length used is

immaterial, provided that the same unit is used for all forces in the solution of any one problem

It is not necessary, in order to represent a force by a line, to draw the line through the point of application, or any other special point. Any line parallel to the line of action of the force may be used for the purpose

86. Vectors.—Any quantity that, like force and velocity, has magnitude and direction is called a **vector quantity;** and the line, as OA, Fig 11, by which the quantity is represented graphically is called a **vector.** A vector is, then, simply a line having a length proportional to the magnitude of a vector quantity, and an arrowhead to indicate the direction of that quantity Such quantities as mass and volume, which have no direction, cannot be represented by vectors.

87. That extremity of a vector to which the arrowhead points is called the **end** of the vector; the other extremity is the **origin.** In Fig. 11, O is the origin and A the end of the vector OA.

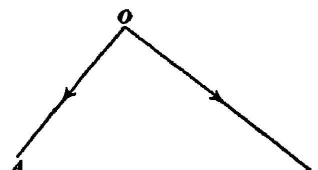


Fig 12

88. Two vectors having the same origin or the same end will here be referred to as being in **non-cyclic order.** If the end of one coincides with the origin of the other, they will be described as being in **cyclic order.** In Fig 12, the vectors OA and OB have the common origin O,

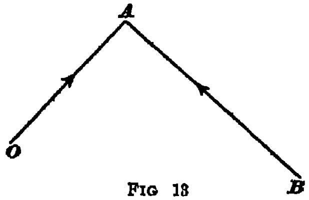


Fig 13

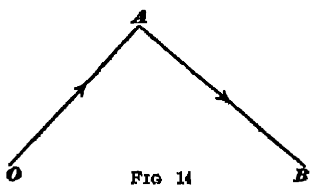


Fig 14

and are, therefore, in non-cyclic order; so are the vectors OA and BA, Fig. 13, which have the common end A. In Fig. 14, the end of the vector OA coincides with the origin of the vector AB, these two vectors are in cyclic order.

89. Resultant and Components.—When a body is acted on by several forces, there is usually one single force that, if acting alone, would produce the same effect as the several forces combined This single force is called the **resultant** of the other forces. With respect to the resultant, the combined forces are called **components.**

90. Composition and Resolution of Forces.—The process of finding the resultant when the components are known is called the **composition of forces.** The process of finding the components when the resultant is known is called the **resolution of forces.**

Both the resolution and the composition of forces are of the utmost importance in dynamics, especially in statics, and will be explained more fully further on. But here the fundamental principles must be stated and explained.

91. Composition of Collinear Forces.—Forces having the same line of action are called **collinear forces.**

The resultant of several collinear forces is equal to their algebraic sum.

For example, if a force of 15 pounds pulls a body upwards, and a force of 10 pounds pulls the body downwards, along the same line, the effect will be the same as if the body were pulled upwards only, with a force of $15 - 10 = 5$ pounds. If the upward direction is treated as positive, the downward direction will be negative, and the downward pull of 10 pounds will be represented by (-10). The resultant is, therefore, $15 + (-10) = +5$. The same principle applies to any number of forces. Thus, the resultant of the forces 15 pounds, 37 pounds, -39 pounds, 13 pounds, and -78 pounds is $15 + 37 - 39 + 13 - 78 = 65 - 117 = -52$ pounds. The negative sign indicates that the resultant is a downward pull of 52 pounds. This principle, which is sufficiently obvious, has already been applied in this Section.

92. Concurrent and Non-Concurrent Forces.—Two or more forces not having the same line of action, but whose lines of action meet at a point, are called **concurrent**

forces. If the lines of action of several forces do not meet at one point, the forces are said to be **non-concurrent.**

93. Resultant of Two Concurrent Forces: Parallelogram of Forces.—Let O, Fig 15, be a particle acted on by two concurrent forces F_1 and F_2 in the directions OA and OB, respectively. These lines are vectors representing the two forces. Complete the parallelogram $OACB$, by drawing AC parallel to OB, and BC parallel to OA. Draw the vector OC, forming the diagonal of the parallelogram. It can be proved, both mathematically and by experiment, that this vector represents the resultant of the forces F_1 and F_2; in other words, that, if the forces F_1 and F_2 are represented in magnitude and direction by the sides OA and OB of a parallelogram, the resultant will be represented in magnitude and direction by the diagonal OC. This principle is called the **law of the parallelogram of forces,** and can be stated in general terms as follows:

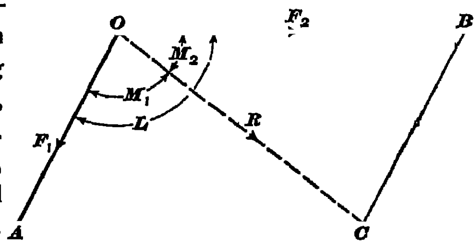


FIG. 15

If two concurrent forces are represented in magnitude and direction by two vectors taken in non-cyclic order, their resultant will be represented, in magnitude and direction, by another vector forming the diagonal of the parallelogram constructed on the vectors representing the two forces.

94. Formulas Derived From the Parallelogram of Forces.—When the magnitudes of the forces F_1 and F_2, Fig. 15, and the angle between their lines of action are given, the magnitude and direction of the resultant R are very readily computed by the principles of trigonometry. In the triangle OAC,

$$OC = \sqrt{\overline{OA}^2 + \overline{AC}^2 - 2\,\overline{OA}.\,\overline{AC}\,\cos A}$$

But $OC = R$, $OA = F_1$, $AC = OB = F_2$, and angle $A = 180° - L$. Substituting these values, and writing

$-\cos L$ for $\cos(180° - L)$, the preceding equation becomes

$$R = \sqrt{F_1^2 + F_2^2 + 2F_1F_2\cos L} \qquad (1)$$

The angle M_1 that R makes with F_1 is found from the same triangle by the principle of sines, which gives $\frac{\sin M_1}{\sin A} = \frac{AC}{OC}$; or, replacing $\sin A$ by $\sin L$ (for $A = 180° - L$), and writing F_2 for AC, and R for OC,

$$\frac{\sin M_1}{\sin L} = \frac{F_2}{R};$$

whence, $$\sin M_1 = \frac{F_2}{R}\sin L \qquad (2)$$

Similarly, $$\sin M_2 = \frac{F_1}{R}\sin L \qquad (3)$$

The values of M_1 and M_2 may be checked by the relation $M_1 + M_2 = L$.

EXAMPLE.—Two concurrent forces having the magnitudes 450 pounds ($= F_1$) and 675 pounds ($= F_2$) make an angle of 60°. Required (*a*) the magnitude R of their resultant, (*b*) the angles M_1 and M_2 made by the line of action of R with the lines of action of F_1 and F_2, respectively.

SOLUTION.—(*a*) Here, $F_1 = 450$ lb., $F_2 = 675$ lb., $L = 60°$, and formula **1** gives

$$R = \sqrt{450^2 + 675^2 + 2 \times 450 \times 675 \times \cos 60°} = 980.75 \text{ lb.} \quad \text{Ans.}$$

(*b*) Substituting in formulas **2** and **3** the values of F_2, F_1, R, and L (angles are given to the nearest 10 seconds),

$$\sin M_1 = \frac{675}{980.75} \times \sin 60°, \text{ whence, } M_1 = 36° 35' 10''. \quad \text{Ans.}$$

$$\sin M_2 = \frac{450}{980.75} \times \sin 60°, \text{ whence, } M_2 = 23° 24' 50''. \quad \text{Ans.}$$

Checking, $M_1 + M_2 = 36° 35' 10'' + 23° 24' 50'' = 60°$.

95. Rectangular Components.—When the two forces are perpendicular to each other, they are called **rectangular components.** In this case, angle $L = 90°$, $\sin L = 1$, $\cos L = 0$, and the preceding formulas become

$$R = \sqrt{F_1^2 + F_2^2} \qquad (1)$$

$$\sin M_1 = \cos M_2 = \frac{F_2}{R} = \frac{F_2}{\sqrt{F_1^2 + F_2^2}} \qquad (2)$$

For this condition, however, the angles M_1 and M_2 are more easily determined by their tangents or cotangents. This special case is illustrated in Fig. 16, which is similar to Fig. 15, except that the angle $A\,O\,B$ is a right angle. The parallelogram $O\,A\,C\,B$ is a rectangle, and the right triangle $O\,A\,C$ gives $R = O\,C = \sqrt{\overline{O\,A}^2 + \overline{A\,C}^2} = \sqrt{F_1^2 + F_2^2}$, as found before. Also,

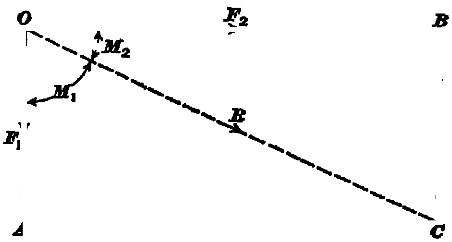


FIG. 16

$$\tan M_1 = \frac{A\,C}{O\,A} = \frac{F_2}{F_1} \qquad (3)$$

Similarly,
$$\tan M_2 = \frac{F_1}{F_2} \qquad (4)$$

96. Resolution of a Force Into Two Components. The next thing to consider is the converse problem of finding the components of a force when the force is given. This is called **resolving** a force into components. We shall here confine ourselves to the particular problem of resolving a force into two components whose directions are given.

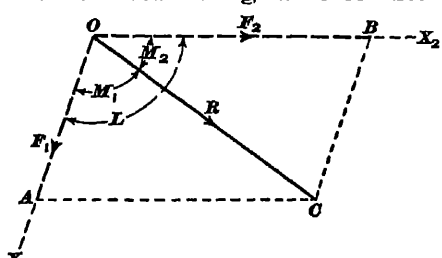


FIG. 17

Let $O\,C = R$, Fig. 17, be a force acting at O. It is required to find its components along two given directions. To do this, draw through O two indefinite lines $O\,X_1$ and $O\,X_2$, parallel to the given directions. From C draw a parallel to $O\,X_2$, meeting $O\,X_1$ at A, and a parallel to $O\,X_1$, meeting $O\,X_2$ at B. Then, $O\,A$ and $O\,B$ are the required components F_1 and F_2.

To find F_1 and F_2 by calculation. Since the directions of F_1 and F_2 are given, M_1 and M_2 are known, and also $L = M_1 + M_2$. Then, from formula **2** of Art. **94**,

$$F_2 = \frac{R \sin M_1}{\sin L} = \frac{R \sin M_1}{\sin (M_1 + M_2)} \qquad (1)$$

Similarly, $F_1 = \dfrac{R \sin M_2}{\sin (M_1 + M_2)}$

If one of the components is to make an angle M_1 with the ›rce R, and the other an angle of $90° - M_1$, that is, if the ›o components are to be perpendicular to each other, for- ula **2** of Art. **95** gives

$$F_2 = R \sin M_1 \qquad (2)$$

Similarly, $F_1 = R \sin M_2 = R \cos M_1 \qquad (3)$

The student should again consider Fig. 16, and see how ›ese formulas are derived. If the figure is kept in mind ›d the geometrical relations of the quantities involved are ›membered, no difficulty in deriving, remembering, and ›plying the formulas will be experienced

97. By *the* component of a force in a certain direction is ›sually understood one of two rectangular components. The ›rce is supposed to be resolved into two components at ›ght angles to each other, one of which has the direction in ›uestion. Such a component is also called the **resolute** and ›e **resolved part** of the force in the given direction.

EXAMPLES FOR PRACTICE

1. If, in Fig 17, OC represents a force of 40 pounds acting on a ›dy at O, and the angles that the components F_1 and F_2 make with ›e given force are 45° and 20°, respectively, what are the values of ›ese components? Ans. $\begin{cases} F_1 = 15.10 \text{ lb.} \\ F_2 = 31.21 \text{ lb.} \end{cases}$

2 A rectangular component F_2 of 60 pounds makes an angle of 40° ›th the resultant R. Find the other component F_1 and the resultant. Ans. $\begin{cases} F_1 = 50.34 \text{ lb.} \\ R = 78.32 \text{ lb.} \end{cases}$

98. The Triangle of Forces.—In the construction ›own in Fig. 17, the complete parallelogram $OACB$ has ›een drawn, in order that the principle on which the construc- ›on is founded may be understood. But, in practice, it is not ›ecessary to draw more than one-half of the parallelogram;

that is, one of the equal triangles OAC, OBC. For, since $OB = AC$, the line CA drawn through C parallel to OX_2 in order to determine the point A, gives at once the length of the line OB representing the component F_2. Also, the angle M_2 between F_2 and R is equal to the angle OCA. Hence the following construction (Fig. 18).

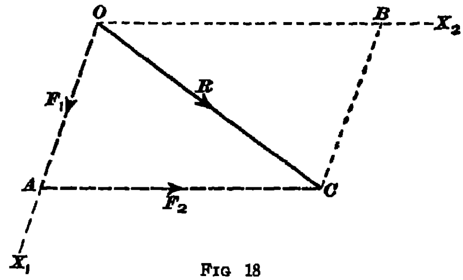


Fig 18

When a force R is given, and it is required to find its components in two given directions, draw from one extremity, as O, of the vector representing the given force, a line parallel to one of the given directions, and from the other extremity, as C, a line parallel to the other given direction The two sides OA and AC of the triangle thus formed are the required components. The vectors representing the two components are in cyclic order with each other, but in non-cyclic order with the vector representing their resultant R.

This construction is called the **triangle of forces.**

Notice that, although AC gives the magnitude and direction of F_2, the force F_2 must be supposed to be really acting at O, as shown by the dotted line OB.

The line CB might have been drawn through C, to meet OX_2 at B In this case, however, the triangle would have been as shown by the light dotted lines, but the results would have been the same as before.

EXAMPLES FOR PRACTICE

1. Two forces, $F_1 = 1{,}000$ pounds and $F_2 = 3{,}000$ pounds, act simultaneously on a particle, the lines of action of the two forces make with each other an angle of 30°. Find (*a*) their resultant R; (*b*) the inclination of the resultant to the two components (angles to nearest 10″).

$$\text{Ans} \begin{cases} (a) & R = 3{,}898.2 \text{ lb.} \\ (b) & \begin{cases} M_1 = 22° \, 37' \, 50'' \\ M_2 = 7° \, 22' \, 10'' \end{cases} \end{cases}$$

2. Given a force $R = 20$ tons, to resolve it into two components F_1 and F_2 making with R angles $M_1 = 30°$, $M_2 = 45°$.

$$\text{Ans. } \begin{cases} F_1 = 14.641 \text{ tons} \\ F_2 = 10.353 \text{ tons} \end{cases}$$

3. Find: (*a*) the resultant of two rectangular forces $F_1 = 40$ pounds and $F_2 = 30$ pounds, (*b*) the inclination of the resultant to the components.

$$\text{Ans. } \begin{cases} (a) \quad R = 50 \text{ lb} \\ (b) \begin{cases} M_1 = 36° \, 52' \, 10'' \\ M_2 = 53° \, 7' \, 50'' \end{cases} \end{cases}$$

4. A force of 100 tons is inclined to the vertical at an angle of 60°. Resolve it into a horizontal component H and a vertical component V.

$$\text{Ans. } \begin{cases} H = 86.603 \text{ tons.} \\ V = 50 \text{ tons.} \end{cases}$$

ANALYTIC STATICS

(PART 1)

CONCURRENT COPLANAR FORCES

DEFINITIONS

1. Coplanar and Non-Coplanar Forces.—When the lines of action of several forces lie in the same plane, the forces are said to be **coplanar**. If the lines of action are not in the same plane, the forces are **non-coplanar**.

2. Analytic Statics and Graphic Statics.—So far as the mathematical treatment of statics is concerned, this science may be considered as being mainly a branch of applied geometry. And, as geometrical problems may be solved either **analytically** or **graphically**—that is, either by computation or by construction—so, too, statics may be either analytic or graphic, according to the method of solution used.

Analytic statics treats of the equilibrium and equivalence of forces by means of the arithmetical and algebraic relations existing among the forces, which relations, in the case of forces represented by vectors, are the same as the relations existing among the vectors, and depend on the geometric properties of the figures formed by combinations of such vectors. In analytic statics, all results are found by calculation.

Graphic statics treats of the equilibrium and equivalence of forces by means of geometric figures. In graphic statics, all results are found by measurement.

3. System of Forces.—The aggregate of all the forces acting on a body or on a group of bodies is called a **system of forces**. The simplest system of forces is obviously that in which there is only one force.

4. Equivalent Systems.—Two or more systems of forces are **equivalent** when they produce the same effect, or, what is the same thing, when they have the same resultant The resultant of any number of forces is itself equivalent to the system formed by the components.

5. Equilibrants.—When a system of forces is balanced by a single force, the latter is called the **equilibrant** of the system Conversely, if several forces balance one single force, they are termed the equilibrants of that force.

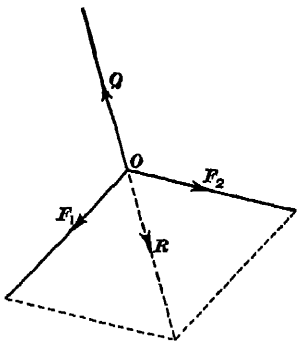


FIG 1

The equilibrant of several forces is evidently equal in magnitude, but opposite in direction, to the resultant of those forces. Hence, it is also called the **anti-resultant.**

In Fig. 1, F_1 and F_2 are two forces acting at O. Their resultant R is found by the principle of the parallelogram of forces. The force Q, equal to R, but acting in the opposite direction, will evidently balance R, or the system of forces F_1 and F_2 to which R is equivalent. Therefore, Q is the equilibrant of F_1 and F_2. Conversely, F_1 and F_2 are the equilibrants of Q.

It is also evident that, if the forces F_1, F_2, and Q are in equilibrium, any of them may be considered as the equilibrant of the two others. The same principle applies to any number of forces.

6. The Resultant Not an Actual Force.—It should be understood that, when several forces act on a body, the resultant is not an actual force acting on the body, but an imaginary force whose effect would be the same as the combined effects of the components. Similarly, when a single

force acts on a body, its components are imaginary forces that would produce the same effect as the single force. The resultant may replace the components and the components may replace the resultant, but in no case are the components and the resultant supposed to act simultaneously.

The equilibrant, on the contrary, is a real force applied in order to balance a system of other forces; or, if the system is balanced, any one of the forces is the equilibrant of the others.

7. Internal and External Forces.—With reference to a body or system of bodies, a force is said to be **external** when it is exerted by a body outside of the system. The forces exerted by parts of a body, or of a system of bodies, on one another are called **internal forces.**

8. The terms "internal" and "external" are relative; and it is evident that a force may be external with respect to a body and internal with respect to another body, or to a system of which the body in question forms a part.

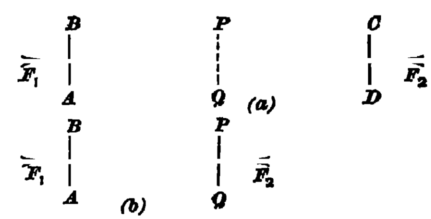


FIG 2

For instance, the forces F_1 and F_2 shown in Fig. 2 (*a*) are external to the body $ABCD$ on which they act. If the body is in equilibrium, any part of it, as $ABPQ$, to the left of a section PQ, must push on the other part with a force equal to F_1, while the other part pushes on it with a force $F_2 = F_1$. This mutual push, or stress, between $ABPQ$ and $CDQP$ consists of two forces, both of which are internal with regard to the whole body $ABCD$. But if the condition of only the part $ABPQ$ is under consideration, the push of $CDQP$ on it may be regarded as an external force. The portion $CDQP$ may be removed, as shown in Fig. 2 (*b*), and the portion $ABPQ$ considered as an independent body acted on by the external forces F_1 and F_2.

FUNDAMENTAL PRINCIPLES AND FORMULAS

9. Free Body: Principle of Separate Equilibrium. When a body is considered by itself as a whole disconnected from other bodies, it is called a **free body.**

Any part of a system or body may be treated as a free body, provided that it is assumed to be under the action of external forces equal to the internal forces that keep it connected with the rest of the body or system of which it forms a part. Thus, in Fig. 2 (*b*), the portion $ABPQ$ may be treated as a free body, after introducing the force F_s, equal to the force exerted by the rest of the body $ABCD$, that is, by $CDQP$, on the portion $ABPQ$.

Fig. 3

As another example, take the system formed by a string OP, Fig 3, carrying a weight W. If it is desired to study the condition of a part of the string, as OA, the part AP and the weight W may be supposed to be removed; but, in order that the condition of OA may remain unaltered, it is necessary to introduce at A an external force equal to the sum of the weight W and the weight of the part AP of the string.

In general, when a body or a system of bodies is in equilibrium, every part of it must be in equilibrium and may be treated as a separate body by itself, provided that external forces are introduced to replace the internal forces exerted by the rest of the body or system on the part removed. This principle is called the **principle of separate equilibrium;** its application, called the **free-body method,** is of great value in the solution of static problems.

10. Transferability of the Point of Application. Concurrent forces have already been defined as those whose lines of action intersect at a point. They may or may not be actually applied at their point of intersection. Their point of application is not essential, provided that the forces act on a rigid body. This is expressed by the following

principle, which is usually known as the principle of the *transmissibility* or *transference of force*, but will here be called the principle of the **transferability of the point of application:**

When a force acts on a rigid body, the force may be supposed to be applied at any point on its line of action, provided that this point and the body are rigidly connected.

Let a force F, Fig. 4, act on the body MN in the direction

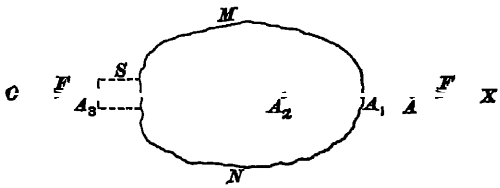


Fig 4

OX. A string may be tied at A_2 or at A_1, and a pull exerted from X with the force F and along the line OX; or it may be imagined that the string is tied to a point A, and that this point is the end of a rigid rod connected to the body at A_1, A_2 or to any other point along OX; no matter where the string is tied, and no matter from what point on OX a pull is exerted, the effect will be the same. Again, it may be imagined that a rigid strut S is attached to the body and that the force is applied at A_3 either by pulling from X or by pushing from O.

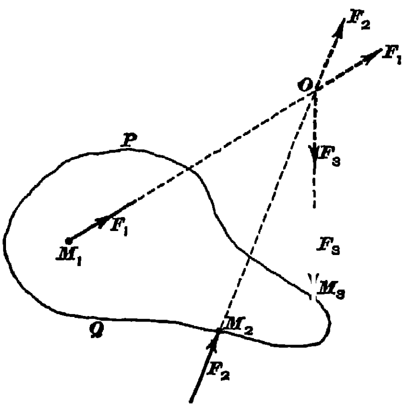


Fig 5

It may thus happen that several concurrent forces acting on a body do not intersect within the body. For example, in Fig 5, the points of application of the forces F_1, F_2, and F_3 actually applied at the points M_1, M_2, and M_3 of the body PQ, may be transferred to the common point of intersection O of their lines of action, assuming this point to be

rigidly connected to the body. Thus, the force F_2 may be imagined to be a pull at the end O of a string tied to M_2, and the force F_3 as a push at the end O of a strut between O and M_3.

11. Resultant of Any Number of Concurrent Forces.—The resultant of two concurrent forces can be easily determined by either the parallelogram or the triangle of forces, as explained in *Fundamental Principles of Mechanics*. The resultant of more than two forces can be found by successive applications of either of these principles.

Let four concurrent forces F_1, F_2, F_3, F_4 be represented, respectively, by the vectors OA, OB, OC, OD, Fig. 6. The resultant R' of F_1 and F_2 is found from the parallelogram $OAEB$ constructed on the vectors OA and OB. This resultant is now combined with another of the given forces, say F_3, by constructing the parallelogram $OEGC$, which gives the resultant R'' of R' and F_3, and, therefore, of F_1, F_2, and F_3. Finally, the parallelogram $OGKD$ gives the resultant R of R'' and F_4, and, therefore, of F_1, F_2, F_3, and F_4.

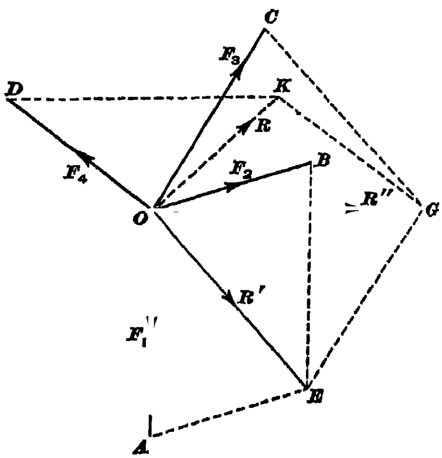


FIG 6

When analytic methods are used, this process is exceedingly laborious, as it involves the calculation of the magnitude and direction of each resultant. A much simpler process is afforded by the resolution of each force into two rectangular components, as explained in the next article. If the problem is solved graphically, the triangle of forces is used instead of the parallelogram: AE is drawn equal and parallel to OB or F_2; then, EG equal and parallel to F_3; then, GK

equal and parallel to F_4, the vector $O K$, between O and K, gives the resultant This method will be further explained and illustrated in connection with graphic statics. At present, only analytic methods are under consideration.

12. Method by Rectangular Components.—Let F_1, F_2, F_3, F_4, Fig. 7, be four concurrent forces represented,

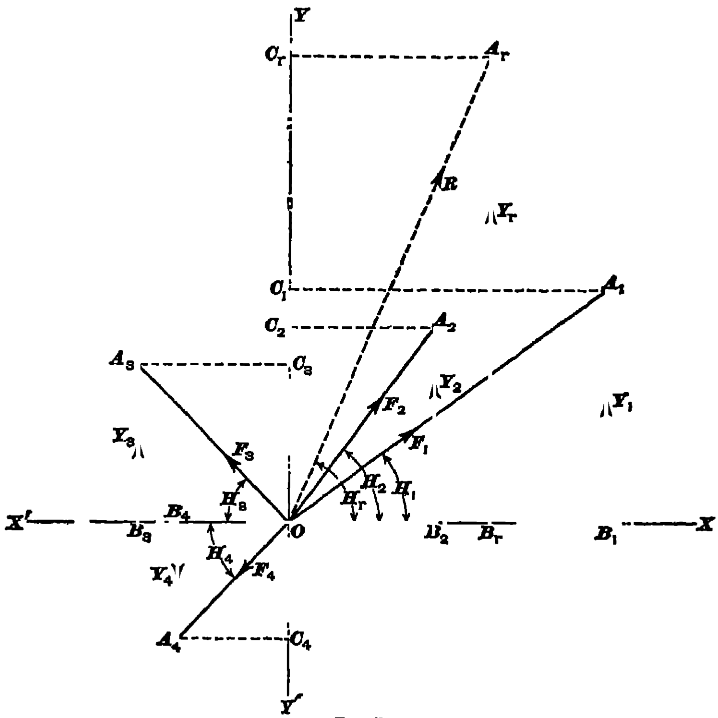


FIG 7

respectively, by the vectors $O A_1$, $O A_2$, $O A_3$, $O A_4$. Take any two lines $X' X$, $Y' Y$ through O and at right angles to each other. These lines are used as lines of reference; they are called **coordinate axes**, and may be taken in any convenient position. The axis $X' X$ is usually referred to as the **x axis**, or **axis of x**, and the axis $Y' Y$, as the **y axis**, or **axis of y**. The angles made by the given forces with the axis of x are denoted by H_1, H_2, etc., as shown.

As explained in *Fundamental Principles of Mechanics*, the force F_1 can be resolved into two components in the directions OX and OY, by drawing from A_1 a line A_1C_1 parallel to OX, and a line A_1B_1 parallel to OY. These lines determine a parallelogram $OB_1A_1C_1$, in which OB_1 and OC_1 represent the components of F_1 in the directions OX and OY, respectively. The component OB_1 is called the **x component** of F_1; and OC_1, the **y component**. These components will be denoted by X_1 and Y_1, respectively. Instead of constructing the parallelogram $OB_1A_1C_1$, it suffices to draw from A_1 the line A_1B_1 perpendicular to OX; this determines the two components of F_1, since $B_1A_1 = OC_1 = Y_1$. The components of F_2, F_3, F_4 are similarly found; they will be denoted by X_2 and Y_2, X_3 and Y_3, etc. It should be understood that, although Y_1, for instance, may be represented by the vector B_1A_1, that component really acts through O, its position being OC_1. The given forces are now reduced to the system formed by the x components OB_1, OB_2, OB_3, OB_4, and the y components OC_1, OC_2, OC_3, OC_4, and the resultant of the given forces is the same as that of these components. Since the x components act in the same line, their resultant is equal to their algebraic sum (see *Fundamental Principles of Mechanics*). The same principle applies to the resultant of the y components. Taking forces acting toward the right or upwards as positive, and those acting toward the left or downwards as negative, we have,

$$X_1 = OB_1,\ X_2 = OB_2,\ X_3 = -OB_3,\ X_4 = -OB_4$$
$$Y_1 = OC_1,\ Y_2 = OC_2,\ Y_3 = \quad OC_3,\ Y_4 = -OC_4$$

Denoting the resultant of the x components and that of the y components by X_r and Y_r, respectively, we have,

$$X_r = X_1 + X_2 + X_3 + X_4 = OB_1 + OB_2 - OB_3 - OB_4 \quad (a)$$
$$Y_r = Y_1 + Y_2 + Y_3 + Y_4 = OC_1 + OC_2 + OC_3 - OC_4 \quad (b)$$

These resultants are represented in the figure by OB_r and OC_r, respectively. The given system of forces has, therefore, been reduced to the two forces $X_r = OB_r$ and $Y_r = OC_r$; and their resultant R, found by the parallelogram or by the triangle of forces, is the resultant of the given forces. In other words, the forces X_r and Y_r are the rectangular

components, in the directions OX and OY, of the required resultant.

13. The calculation of the magnitude and direction of R is accomplished as follows. The right triangle OB_1A_1 gives:

$$OB_1 = OA_1 \cos H_1, \; B_1A_1 = OA_1 \sin H_1,$$

or, since $OB_1 = X_1$, $OA_1 = F_1$, and $B_1A_1 = Y_1$,

$$X_1 = F_1 \cos H_1, \; Y_1 = F_1 \sin H_1$$

The values of X_2 and Y_2, X_3 and Y_3, and X_4 and Y_4 are found in the same manner It may be well to remember the principle that *the component or resolute of a force in any direction is equal to the force multiplied by the cosine of the acute angle that the line of action of the force makes with that direction.* Thus, the x component of F_2 is $F_2 \cos H_2$ For the other component, the sine should be used instead of the cosine

Substituting in equations (a) and (b) of the preceding article the values of X_1, Y_1, etc., the following expressions are obtained:

$$X_r = F_1 \cos H_1 + F_2 \cos H_2 - F_3 \cos H_3 - F_4 \cos H_4$$

$$Y_r = F_1 \sin H_1 + F_2 \sin H_2 + F_3 \sin H_3 - F_4 \sin H_4$$

Having determined X_r and Y_r, the right triangle OB_rA_r gives the magnitude of the resultant R, and the inclination H_r of this resultant to OX.

$$R = OA_r = \sqrt{OB_r^2 + B_rA_r^2} = \sqrt{X_r^2 + Y_r^2} \qquad (1)$$

$$\tan H_r = \frac{B_rA_r}{OB_r} = \frac{Y_r}{X_r} \qquad (2)$$

14. As explained in *Plane Trigonometry*, Part 2, the symbol of summation Σ, read *sigma*, is often used to indicate the algebraic addition of several quantities denoted by the one letter affected by subscripts or accents. If, for example, several quantities are denoted by X_1, X_2, X_3, X_4, etc., their algebraic sum $X_1 + X_2 + X_3 + X_4 \; . \; . \; .$ is denoted by the expression ΣX, read *sigma x*.

With this notation, the algebraic sum of the x components X_1, X_2, etc. of any number of forces may be denoted by ΣX, and the algebraic sum of the y components by ΣY. Also, the sum of the expressions $F_1 \cos H_1, F_2 \cos H_2$, etc. may be denoted by $\Sigma F \cos H$. If the components of the resultant

are, as before, denoted by X_r and Y_r, the following equations apply to any number of forces

$$X_r = \Sigma X = \Sigma F \cos H \qquad (1)$$
$$Y_r = \Sigma Y = \Sigma F \sin H \qquad (2)$$

Having computed X_r and Y_r, the magnitude and direction of the resultant R are determined by formulas **1** and **2** of Art. **13.**

EXAMPLE —It is required to find the resultant of the three con-

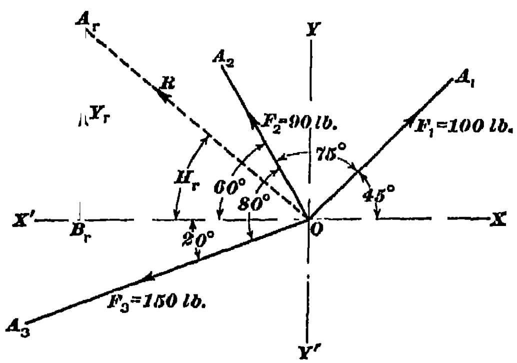


FIG. 8

current forces F_1, F_2, F_3 represented in Fig 8, their magnitudes and relative positions being as shown

SOLUTION —Take $X'X$ inclined at 45° to the line of action of F_1. The inclinations of the other two forces to $X'X$ are readily determined from the given angles. Thus,

$$H_1 = 45°,\ H_2 = X'OA_2 = 180° - (75° + 45°) = 60°$$
$$H_3 = X'OA_3 = 80° - 60° = 20°$$

For the x components, with the usual convention as to signs, the following equations are obtained:

$$X_1 = 100 \cos 45° = 70.711 \text{ lb.}$$
$$X_2 = -90 \cos 60° = -45.000 \text{ lb.}$$
$$X_3 = -150 \cos 20° = -140.95 \text{ lb.}$$

and for the y components:

$$Y_1 = 100 \sin 45° = 70.711 \text{ lb.}$$
$$Y_2 = 90 \sin 60° = 77.943 \text{ lb.}$$
$$Y_3 = -150 \sin 20° = -51.303 \text{ lb.}$$

Therefore,

$$X_r = \Sigma X = 70.711 - 45.000 - 140.95 = -115.24 \text{ lb.}$$
$$Y_r = \Sigma Y = 70.711 + 77.943 - 51.303 = 97.351 \text{ lb.}$$

The negative sign of X_r indicates that this component of the resultant R is directed toward the left, as indicated by the vector OB_r. Formula 1 of Art 13 now gives,

$$R = \sqrt{115.24^2 + 97.351^2} = 150.86 \text{ lb.}$$

and formula 2 of the same article gives,

$$\tan H_r = \frac{97.351}{115.24}, \quad H_r = 40° 11' 20''$$

In finding the value of tan H_r, the negative sign of X_r is disregarded, as the only thing that is required is the numerical value of the angle. The signs of X_r and Y_r show in what quadrant the resultant is. In the present case, X_r $(= OB_r)$ is negative and Y_r $(= B_r A_r)$ is positive. This at once shows that R lies in the angle $X' O Y$, and that H_r is, therefore, the angle that R makes with OX'.

NOTE.—In this, as in many other problems in this Course, angles are given to the nearest 10''. Thus, an angle of 35° 16' 37'' is called 35° 16' 40'', an angle of 17° 47' 23'' is called 17° 47' 20''.

MOMENTS

15. Definitions.—The **moment** of a force **about a point,** or **with respect to a point,** is the product obtained by multiplying the magnitude of the force by the perpendicular distance from the point to the line of action of the force. In Fig. 9, the moment of F about the point C is Fp; about the point C_1, it is Fp_1.

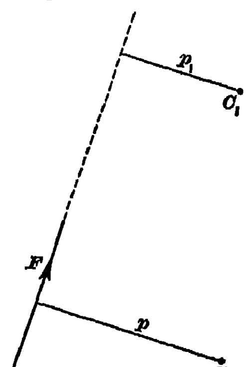


FIG. 9

16. The point to which a moment is referred, or about which a moment is taken, is called the **center of moments,** or **origin of moments.** The perpendicular p or p_1 from the origin on the line of action of the force is called the **lever arm,** or simply the **arm,** of the force with respect to the origin.

17. A moment is expressed in foot-pounds, inch-tons, etc., according to the units to which the force and its arm are referred. If, for instance, the force is 10 pounds and the arm is 60 feet, the moment is 10 × 60, or 600, foot-pounds; if the force is 8 tons and the arm is 6 inches, the moment is

8 × 6, or 48, inch-tons. The term *foot-pound* is used in kinetics in another sense; but the two meanings are so different that there is no danger of confusion

18. Sign and Direction of a Moment.—When, looking in the direction of the arrowhead of a vector representing a force, the center of moments lies on the right of the line of sight, as C and C_1, Fig. 10, the moment is considered positive; if the center is at the left, as C_2, the moment is considered negative This may be stated otherwise thus: Imagine the lever arm p to be revolving about C in the direction indicated by the arrowhead. In this case, p will revolve from left to right, or clockwise, and the moment is considered positive. In the case of p_2, on the contrary, if this line revolved about C_2, following the direction of the arrowhead, its motion would be counter-clockwise; the moment is then considered negative. The direction of the motion just referred to is called the **direction of the moment**, and is supposed to have the same sign as the moment itself.

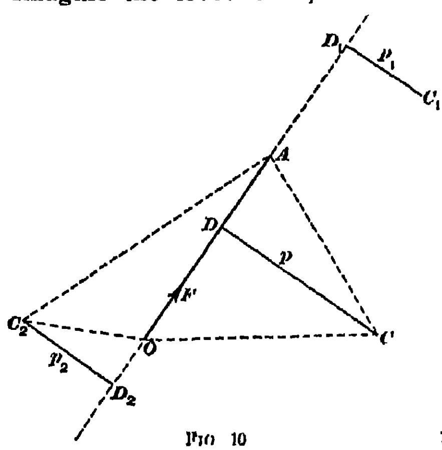


FIG 10

If M, M_1, and M_2 are the moments of F about C, C_1, and C_2, respectively, then,

$$M = +Fp = Fp$$
$$M_1 = +Fp_1 = Fp_1$$
$$M_2 = -Fp_2$$

19. Representation of a Moment by an Area.—If the lines CO and CA, Fig. 10, are drawn, the area of the triangle COA is

$$\tfrac{1}{2}\,OA \times CD = \tfrac{1}{2}\,Fp = \tfrac{1}{2}\,M$$

whence, $M = 2(\tfrac{1}{2}\,OA \times CD) = 2 \times \text{area } OAC$

The magnitude of the moment M is, therefore, numerically equal to twice the area of the triangle whose base is the vector representing the force F and whose opposite vertex is the origin of moments.

20. Moment of a Force in Terms of the Moments of Its Components.—*The moment of a force about any point in its plane is equal to the algebraic sum of the moments of its components in any directions.*

Let the force R, Fig. 11, be resolved into the two components F_1 and F_2 in any directions, and let C be the origin

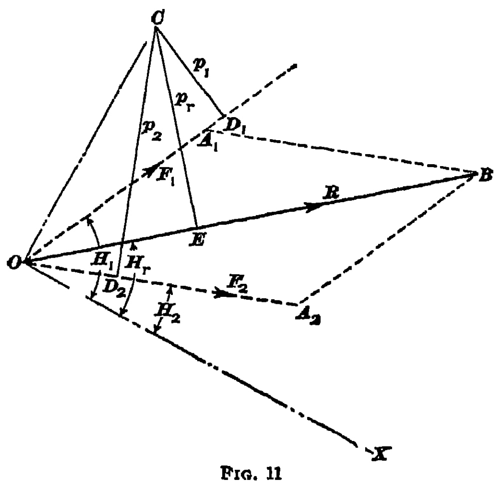


Fig. 11

of moments. The arms of R, F_1, and F_2 are, respectively, p_r, p_1, and p_2, as shown. The moments of these three forces about C will be denoted by M_r, M_1, and M_2, respectively. Draw CO, and OX perpendicular to CO. Denote the angles made by the lines of action of the three forces with OX by H_r, H_1, H_2, as shown, and the resolutes of the forces in the direction OX by X_r, X_1, and X_2. Then (Art. 12),

$$X_r = X_1 + X_2;$$

that is (Art. 13),

$$R \cos H_r = F_1 \cos H_1 + F_2 \cos H_2;$$

whence, multiplying both members of this equation by CO,

$$R \times CO \cos H_r = F_1 \times CO \cos H_1 + F_2 \times CO \cos H_2 \quad (a)$$

But $H_r = 90° - COE$, and, therefore, $\cos H_r = \sin COE$; and

$$CO \cos H_r = CO \sin COE$$
$$= CE \text{ (triangle } COE) = p_r$$

Also,
$$CO \cos H_1 = CO \sin COD_1 = p_1$$
$$CO \cos H_2 = CO \sin COD_2 = p_2$$

Substituting these values in (a),

$$Rp_r = F_1 p_1 + F_2 p_2;$$

that is,
$$M_r = M_1 + M_2.$$

In general, if the algebraic sum of the moments of any number of concurrent forces is denoted by ΣM, and the moment of the resultant by M_r, as above, then,

$$M_r = \Sigma M$$

CONDITIONS OF EQUILIBRIUM

21. The Absolute Condition.—When any number of concurrent forces are in equilibrium, it is evident that they can have no resultant; that is, if their resultant is R, we must have $R = 0$. This is the general condition, sometimes called the **absolute condition**, of equilibrium. Expressed in the form of $R = 0$, however, this condition is of no value in the solution of problems. It is necessary to express R in terms of other quantities, or to derive some other relations by means of which unknown quantities can be determined.

22. Condition of Resolutes.—Since the resultant of several forces in equilibrium is zero, its resolutes in any two directions must be zero; and, as each resolute is equal to the algebraic sum of the corresponding resolutes of the given forces, it follows that *the sum of the resolutes of the forces in any direction must be zero.*

The resolute of a force in a direction perpendicular to itself is zero. There is, then, one direction (at right angles to the resultant) along which the sum of the resolutes of any number of unbalanced concurrent forces is zero; but if the sum of the resolutes is zero for any two directions, the forces are in equilibrium; for both directions cannot be perpendicular to the line of action of the resultant.

23. Condition of Moments.—Since the resultant is 0, its moment about any point must be zero; and, as the moment of the resultant is equal to the sum of the moments of the components, we must have,

$$M_1 + M_2 + M_3 + \quad . \quad . \quad . \quad = 0,$$

where, M_1, M_2, M_3, etc. are the moments of the components about any point in their plane.

It is necessary to specify that the sum of the moments must be zero, when taken with respect to *any* point, or *every* point in their plane, for, in a system of forces not in equilibrium, the sum of their moments about a point on the resultant is also equal to zero, because in this case the lever arm of the resultant is zero, and its moment, therefore, is zero. But, if the forces have a resultant, the sum of their moments about a point outside of the resultant cannot be zero. Conversely, if the sum of the moments of the forces about three points not in the same straight line is zero, the forces have no resultant; for these three points cannot all lie on the line of action of the resultant, and the only case in which the sum of the moments of a system of unbalanced forces can be zero is that in which the origin of moments is a point in the line of action of the resultant.

24. General Statement of the Conditions of Equilibrium.—Summing up: Any balanced system of coplanar concurrent forces must satisfy the following conditions, each of which is necessary and sufficient for equilibrium, and involves the other two:

1. *The resultant of the forces must be zero.*

2. *The sum of the resolutes of the forces in each of any two directions must be zero.*

3. *The sum of the moments of the forces about three points in their plane, not in the same straight line, must be zero.*

Condition 1 is expressed algebraically by the equation $R = 0$.

Condition 2 may be expressed thus (see Art. 14):

$$\left.\begin{aligned} \Sigma X &= \Sigma F \cos H = 0 \\ \Sigma Y &= \Sigma F \sin H = 0 \end{aligned}\right\} \qquad (1)$$

Condition 3 may be expressed thus (see Art. 20):

$$\Sigma M = 0 \qquad (2)$$

If the lever arms of the forces F_1, F_2, etc are denoted by p_1, p_2, etc., the algebraic sum of their moments, or ΣM, is equal to $F_1 p_1 + F_2 p_2 + F_3 p_3 +$, etc Denoting this sum by ΣFp, formula 2 may be written

$$\Sigma Fp = 0 \qquad (3)$$

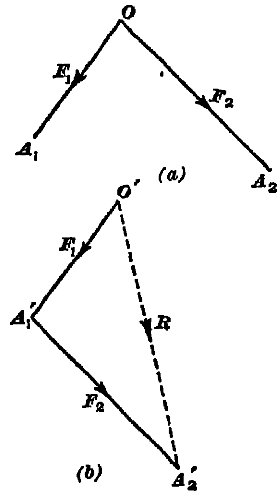


FIG 12

25. Equilibrium of Three Forces.—Problems relating to the equilibrium of three forces are usually most conveniently solved by means of the triangle of forces. It is to be observed that the equilibrant of any number of forces is numerically equal to their resultant, but acts in the opposite direction. Let the forces F_1, F_2, Fig. 12 (a), act through O. By drawing through O', as shown at (b), the vectors $O' A_1'$ and $A_1' A_2'$, representing the given forces, and completing the triangle $O' A_1' A_2'$, the resultant R of F_1 and F_2 is obtained. In this triangle of forces, the two components are in cyclic order with each other, but in non-cyclic order with the resultant (see *Fundamental Principles of Mechanics*).

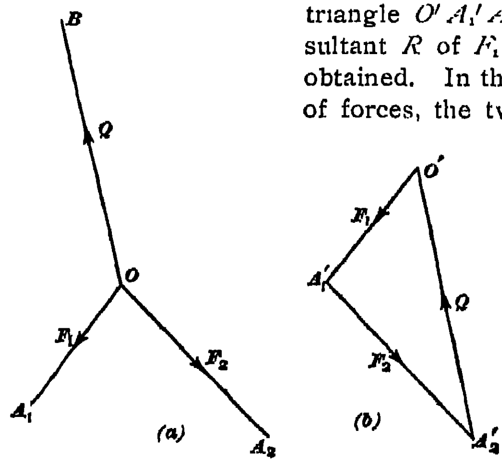


FIG 13

In Fig. 13 (*a*), the three forces F_1, F_2, and Q, acting at O, are supposed to be in equilibrium. The force Q is the equilibrant of F_1 and F_2; it is numerically equal to their resultant R, but acts in the opposite direction. Consequently, a triangle $O' A_1' A_2'$, Fig. 13 (*b*), equal to the triangle giving the resultant R, can be formed with the vectors representing F_1, F_2, and Q; but the arrowheads on Q and R must point in opposite directions, which means that Q must be taken in cyclic order with F_1 and F_2.

26. Selection of Axes.—When there are more than three forces in equilibrium, the most convenient method for determining any of them, when the others are known, consists in finding expressions for the resolutes of all the forces in two rectangular directions, and using formulas **1** and **2**, Art **24**. A similar method applies to the determination of the resultant, as already explained Theoretically these directions are entirely arbitrary; but, practically, the direction of one of the forces, often one whose magnitude is not known, is almost always a very convenient one to use, as in this case one of the resolutes of that force is equal to zero, and the other is equal to the force itself. When this is done, either direction along the line of action of the force may be taken as positive and the other as negative; but this does not necessarily mean that the direction taken as positive is the direction of the force acting along that line. For example, forces acting at a point O may be resolved into components parallel and perpendicular to the line of action, say OK, of one of the forces; and, for the purpose of this resolution, the direction OK may be treated as positive and the opposite direction as negative. This simply means that, if the resolutes parallel to OK are to be added algebraically, the arithmetical difference must be taken between those resolutes whose direction is OK and those whose direction is opposite; if the difference is positive, the direction of the resultant resolute is from O toward K; if negative, from K toward O.

Similarly, in taking moments, any point can be used as an origin; but it is convenient to take the origin on the line or

action of one of the forces, as in this case the moment of that force is zero, and the force is thereby eliminated from the equation of moments.

These methods will be better understood by a study of the following problems:

EXAMPLE 1.—A weight of 500 pounds (see Fig. 14) hangs by three

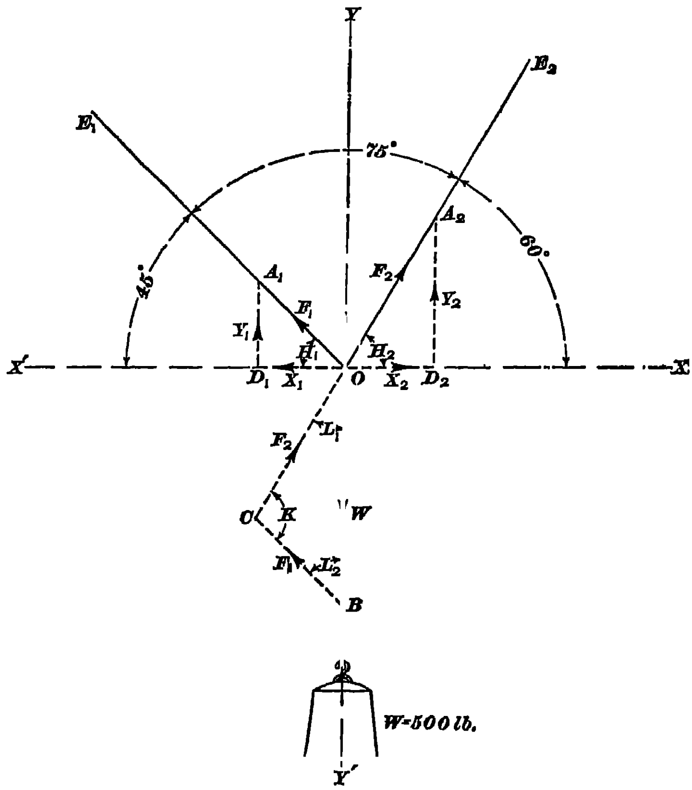


FIG. 14

ropes tied to a ring at O, the inclinations of the two slanting ropes $O E_1$ and $O E_2$ to each other and to the horizontal being as shown. It is required to find the tensions in the ropes $O E_1$ and $O E_2$.

SOLUTION BY TRIANGLE OF FORCES.—The tensions F_1 and F_2 in the ropes are represented by the vectors $O A_1$ and $O A_2$, and the weight by the vector W. These three forces form a balanced system, each force being the equilibrant of the other two. Considering W as the

equilibrant of F_1 and F_2, the triangle of forces OBC is obtained by drawing through B a line parallel to OA_1, and through O a line parallel to OA_2. In the present case, CO is in the prolongation of OA_2. The triangle might have been constructed anywhere else; it is not necessary to draw it in the position here shown. Nor is it necessary to draw it accurately to scale, as its only object is to serve as a guide in the calculation.

In this triangle,

$$K = A_1OC = 180° - A_1OA_2 = 180° - 75°;$$

and, therefore,

$$\sin K = \sin (180° - 75°) = \sin 75°$$

Also, OY being vertical,

$$L_1 = 90° - H_2 = 90° - 60° = 30°$$
$$L_2 = A_1OY = 90° - H_1 = 90° - 45° = 45°$$

The triangle OBC now gives

$$F_1 = \frac{W}{\sin K} \sin L_1 = \frac{500}{\sin 75°} \sin 30° = 258.82 \text{ lb.} \quad \text{Ans.}$$

$$F_2 = \frac{W}{\sin K} \sin L_2 = \frac{500}{\sin 75°} \sin 45° = 366.03 \text{ lb.} \quad \text{Ans.}$$

SOLUTION BY RESOLUTES.—Let the vertical line OY and the horizontal line OX be taken as axes of coordinates. As usual, the resolutes of F_1 will be denoted by X_1 and Y_1, and those of F_2 by X_2 and Y_2, as shown in the figure. They are here drawn for purposes of illustration, but it is not necessary to draw them in order to apply the general formulas. The vertical resolute of the weight is $-W$; the horizontal resolute is zero.

Placing the sum of the horizontal resolutes equal to zero, we have, noticing that X_1 is negative,

$$-F_1 \cos H_1 + F_2 \cos H_2 + 0 = 0;$$

that is,

$$-F_1 \cos 45° + F_2 \cos 60° = 0 \qquad (a)$$

Similarly, for the vertical resolutes,

$$F_1 \sin H_1 + F_2 \sin H_2 - W = 0;$$

or,

$$F_1 \sin 45° + F_2 \sin 60° - W = 0 \qquad (b)$$

From (a),

$$F_2 = F_1 \frac{\cos 45°}{\cos 60°} \qquad (c)$$

which, substituted in (b), gives

$$F_1 \sin 45° + F_1 \cos 45° \frac{\sin 60°}{\cos 60°} - W = 0;$$

whence,

$$F_1 = \frac{W}{\sin 45° + \cos 45° \dfrac{\sin 60°}{\cos 60°}} = \frac{W \cos 60°}{\sin 45° \cos 60° + \cos 45° \sin 60°}$$

$$= \frac{W \sin 30°}{\sin (45° + 60°)} = \frac{W}{\sin 105°} \sin 30° = \frac{W}{\sin 75°} \sin 30°,$$

as found before.

The value of F_2 may now be found from (c).

The horizontal and vertical resolutes have been used simply to illustrate the general method, but it will be seen that, by resolving the forces in these directions, the solution is long and tedious If however, resolutes perpendicular to the lines of action of the unknown forces F_1 and F_2 are taken, the solution will be very much simplified First, let the resolutes be taken perpendicular to $O E_2$ Then, we shall have, considering the direction from O toward E_2 as positive,

$$\text{resolute of } F_2 = 0$$
$$\text{resolute of } F_1 = F_1 \sin 75°$$
$$\text{resolute of } W = -W \sin L_1 = -W \sin 30°$$

Therefore (formula **1**, Art **24**),

$$F_1 \sin 75° - W \sin 30° = 0$$

whence,

$$F_1 = \frac{W}{\sin 75°} \sin 30°$$

In a similar manner, F_2 may be found by taking the resolutes perpendicular to F_1

This method should be carefully studied, as its simplicity makes it of very great practical importance.

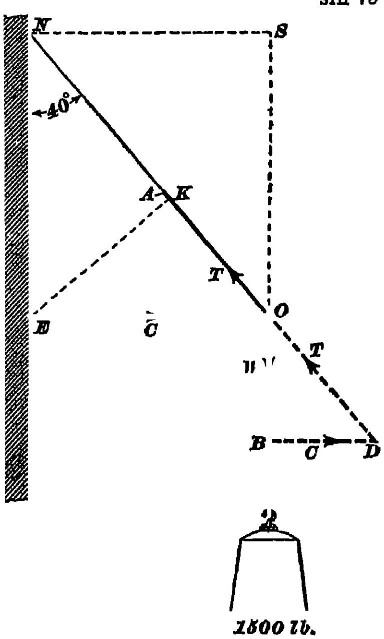


FIG 15

EXAMPLE 2.—A weight W of 1,500 pounds is hung from the extremity O, Fig 15, of a horizontal bar projecting out of a vertical wall and held by a rope $O N$, inclined to the vertical at an angle of 40° Required the tension T in the rope and the compression C in the bar

SOLUTION BY TRIANGLE OF FORCES.—It is obvious that W is the equilibrant of C and T. The actual lines of action of W and T are shown by the vectors $O B$ and $O A$. Draw $B D$ perpendicular to $O B$, to meet $N O$ produced at D. Then, in the triangle $O B D$, the vectors $D O$ and $B D$ represent T and C, respectively. Also, $B O D = 40°$.

$$C = W \tan 40° = 1{,}500 \tan 40° = 1{,}258.6 \text{ lb. Ans}$$

$$T = \frac{W}{\cos 40°} = \frac{1{,}500}{\cos 40°} = 1{,}958.1 \text{ lb Ans.}$$

SOLUTION BY MOMENTS.—If moments are taken about N, the moment of T will evidently be zero, since N lies on the line of action

of T, the moment of W will be $W \times NS = 1{,}500 \times EO$ The moment of C, whose line of action is OE, will be $C \times EN$ Therefore (Art 20),

$$C \times EN = 1{,}500 \times EO,$$

whence, $$C = 1{,}500 \frac{EO}{EN} = 1{,}500 \tan 40°,$$

as found before

If moments are taken about E, the moment of C, the resultant, will be zero, that of W will be the same as before, or $1{,}500 \times EO$, and that of T will be $-T \times EK$—negative, because the origin E is on the

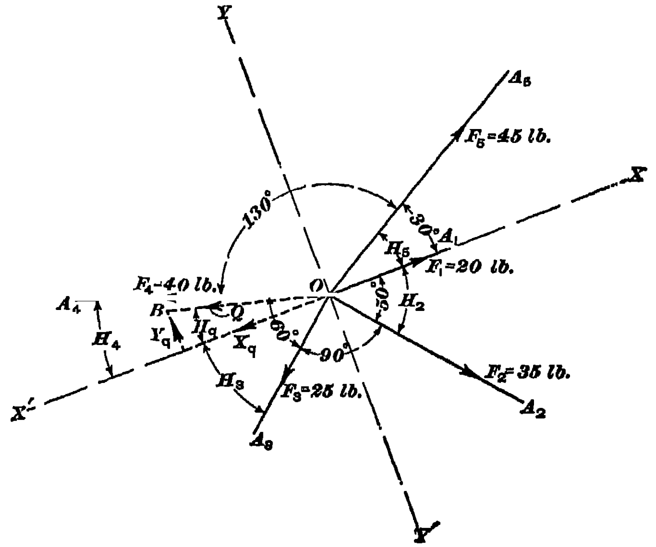


FIG 16

left of OA (Art. 18) Equating the sum of the moments of the components to the moment of the resultant,

$$1{,}500 \times EO - T \times EK = 0,$$

whence, $$T = 1{,}500 \frac{EO}{EK} = \frac{1{,}500}{\sin EON} = \frac{1{,}500}{\cos 40°},$$

as found before

The method of moments is sometimes very convenient. The center of moments should, if possible, be taken on the line of action of one of the unknown forces, as that force is thereby eliminated

EXAMPLE 3 —To find the equilibrant of the five forces represented by the full-line vectors in Fig. 16.

SOLUTION —Let OA_1, the line of action of F_1, be taken as the x direction, and OY, the perpendicular to it at O, as the y direction of

resolutes. The angles of F_1, F_2, F_3, etc, with OX will be denoted by H_1, H_2, H_3, as in Art **12**, and the resolutes of the forces by X_1, X_2, Y_1, Y_2, etc. Here we have $H_1 = 0$, $H_2 = 50°$, $H_3 = A_3OX'$ $= 180° - (H_2 + 90°) = 180° - 140° = 40°$, $H_4 = 60° - H_3 = 60° - 40°$ $= 20°$, $H_5 = 30°$.

Therefore,

$$\begin{aligned} X_1 &= F_1 = 20 \\ X_2 &= F_2 \cos 50° = 35 \cos 50° \\ X_3 &= -F_3 \cos 40° = -25 \cos 40° \\ X_4 &= -F_4 \cos 20° = -40 \cos 20° \\ X_5 &= F_5 \cos 30° = 45 \cos 30° \end{aligned}$$

Denoting the x resolute of the equilibrant by X_q, we must have (Art **24**),

$$X_q + X_1 + X_2 + X_3 + X_4 + X_5 = 0$$

whence,

$$\begin{aligned} X_q &= -(X_1 + X_2 + X_3 + X_4 + X_5) \\ &= -20 - 35 \cos 50° + 25 \cos 40° + 40 \cos 20° - 45 \cos 30° \\ &= -24.730 \text{ lb} \end{aligned}$$

Similarly,

$$\begin{aligned} Y_1 &= 0 \\ Y_2 &= -F_2 \sin 50° = -35 \sin 50° \\ Y_3 &= -F_3 \sin 40° = -25 \sin 40° \\ Y_4 &= F_4 \sin 20° = 40 \sin 20° \\ Y_5 &= F_5 \sin 30° = 45 \sin 30° \end{aligned}$$

and

$$\begin{aligned} Y_q &= -(Y_1 + Y_2 + Y_3 + Y_4 + Y_5) \\ &= 35 \sin 50° + 25 \sin 40° - 40 \sin 20° - 45 \sin 30° \\ &= 6.700 \text{ lb} \end{aligned}$$

The magnitude of the equilibrant Q may be found from the relation $Q = \sqrt{X_q^2 + Y_q^2}$, but it is better to determine first the angle H_q between Q and OX. For this purpose, it is not necessary to take the sign of X_q into account (Art **14**)

$$\tan H_q = \frac{Y_q}{X_q} = \frac{6.700}{24.730}; \quad H_q = 15° \, 9' \, 30''$$

and, therefore,

$$Q = \frac{Y_q}{\sin H_q} = \frac{6.700}{\sin 15° \, 9' \, 30''} = 25.622 \text{ lb. Ans.}$$

EXAMPLES FOR PRACTICE

1 Two concurrent forces $F_1 = 30$ tons and $F_2 = 25$ tons act at an angle of 170°. Find (*a*) the magnitude Q of their equilibrant; (*b*) the angle L made by Q with F_2.

Ans $\begin{cases} (a) \; Q = 6.9132 \text{ tons} \\ (b) \; L = 48° \, 54' \end{cases}$

2. Resolve a force $F = 100$ pounds into two equal components $F_1 = F_2$ making an angle of 70°. Ans. $F_1 = F_2 = 61.039$ lb.

3. Find a general relation between a force F and either of two equal components F_1 making an angle K with each other.

Ans. $F = 2 F_1 \cos \frac{1}{2} K$

4 Given the equilibrant $Q = 10$ tons of the two forces F_1 and F_2, Fig 17, find the magnitudes of those two forces, the angles being as shown (The vectors F_1 and F_2 in the figure are drawn of arbitrary lengths, their purpose being to indicate directions, not magnitudes, the latter being as yet unknown)

Ans $\begin{cases} F_1 = 11\ 154 \text{ T.} \\ F_2 = 2\ 9886 \text{ T.} \end{cases}$

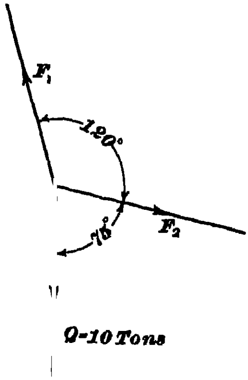


FIG 17

5 Find the equilibrant Q of the forces represented in Fig 18, and the angle H_q it makes with the line of action OX of the 20-ton force (Find resolutes parallel and perpendicular to OX)

Ans $\begin{cases} Q = 10\ 735 \text{ tons} \\ H_q = 5°\ 15'\ 50'' \end{cases}$

6 A weight of 15 tons is supported by two ropes, one horizontal and the other making an angle of 45° with the horizon (135° with the former rope) Find the tensions T_1 and T_2 in the two ropes, T_1 being tension in horizontal rope.

Ans $\begin{cases} T_1 = 15 \text{ tons} \\ T_2 = 21.213 \text{ tons} \end{cases}$

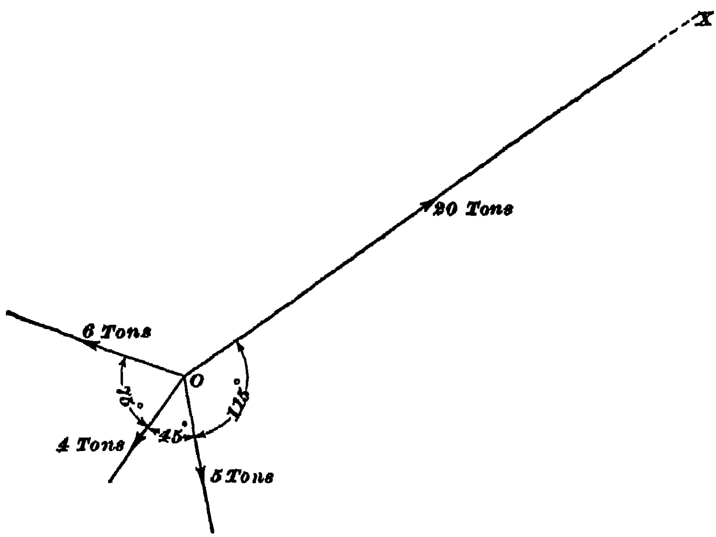


FIG 18

STRESSES IN FRAMED STRUCTURES

DEFINITIONS

NOTE —The theory of concurrent forces finds some of its principal applications in the determination of stresses in framed structures Although a complete treatment of such a subject would be foreign to the scope of this Section, a few simple examples will be given, in order that the utility of these principles may be seen Before giving those examples, it will be necessary to define some of the terms employed

27. Structures. A **structure** is a statical combination of parts designed for the transmission of force. By saying that a structure is a *statical* combination, it is meant that the forces transmitted by the parts are supposed to be all balanced and produce no motion. Bridges and buildings are examples

28. Machines.—A **machine** is a combination of parts designed for the transmission of motion In any special case in which a machine produces no motion, its various parts being at rest, the machine is to be treated as a structure.

29. Frames.—The term **frame** is applied to any combination of bars, strings, ropes, or other straight parts connected together so that their center lines form a polygon or a part of a polygon. Every one of the straight parts so connected is called a **member** of the frame, and the connection of two or more members is called a **joint.**

Although in some cases the center lines of the various members connecting at a joint do not meet exactly in a point, they are considered so to meet, and the forces acting along the members are treated as concurrent forces meeting at the center of the joint.

30. Trusses.—A **truss** is a rigid frame consisting of triangles, such as occur in bridges and buildings, and in some branches of carpentry.

31. Supports.—The place or places on which a structure rests, and to which, therefore, the forces acting on the structure considered as a whole are transmitted, are called

supports, because they *support*, or sustain, the structure. Such are the piers of a bridge and the foundation of an engine.

When a structure, as a bridge, rests on different supports, the point of application of the resultant pressure on any support is called a **point of support.**

32. Reactions.—At every joint of a frame there are several forces acting, namely:

1 The **external forces,** or forces applied to the frame from the outside

2 The **mutual reactions,** or the forces exerted by the various members on one another at the joint.

At some joints, there may be no external forces directly applied, but the members may act on one another on account of external forces acting at other joints.

As to every action there is an equal and opposite reaction, every support of a structure exerts on the structure a force equal and opposite to the force transmitted by the structure to the support. This force exerted by the support on the structure is called the **reaction** at the support, and is to be considered as one of the external forces acting on the structure.

33. Struts and Ties.—The two kinds of stress—tension and compression—that may occur in the members of a frame are defined in *Fundamental Principles of Mechanics*. A tension is often called a **pull,** and a compression a **thrust.**

Those members that are designed to resist compression are called **struts;** those that are designed to resist tension are called **ties.**

DETERMINATION OF STRESSES

34. Introductory Explanations.—Here those frames only will be dealt with in which the forces acting on every member are applied at its extremities, and in which the conditions are such that the mutual reactions at the joints may be taken to have lines of action meeting at the common intersection of the center lines of the members.

This being understood, let A_1A, A_2A, A_3A, Fig. 19, be three members of a frame, A being the point of intersection

of the center lines of the members, and A_1, A_2, A_3, the joints connecting the three members with other members. It will be first assumed that there is no external force acting at A. Let F_1, F_2, F_3 be the resultants of the forces acting on A_1A, A_2A, and A_3A at A_1, A_2, and A_3, respectively. By the principle of separate equilibrium (Art. 9), every member, considered as a free body, must be in equilibrium; therefore, the resultant of the forces acting at A on A_1A must balance F_1, and that resultant and F_1 must be collinear and act along AA_1; so that the resultant at A is a force $-F_1$ equal

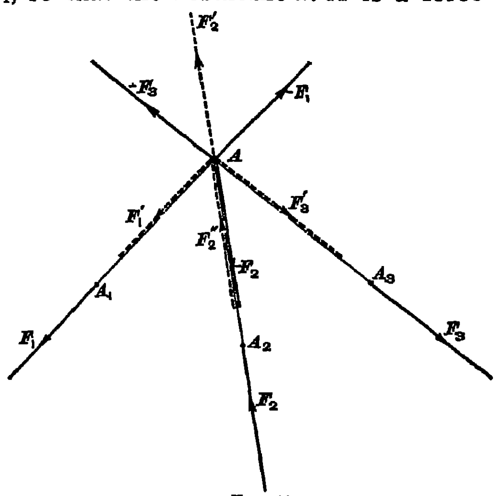


FIG. 19

in magnitude to F_1, but acting in an opposite direction. Likewise, AA_2 is kept in equilibrium by the forces F_2 and $-F_2$, both having the direction of the member, and $-F_2$ being the resultant of the forces exerted by the other members on A_2A at A. Similarly for A_3A. The vector representing $-F_2$ is drawn on one side of the member, in order to avoid confusion, although $-F_2$ really acts along the member AA_2. Now, transfer the forces F_1, F_2, and F_3 to A (Art 10), where they are represented by accented letters. For convenience, as well as for the sake of uniformity, the forces thus transferred are represented by vectors F_1', F_2', F_3', having their common origin at A—that is,

with the arrowheads pointing away from A. It is evident that the force $-F_1$, being the resultant of the forces acting directly on A_1A at A, is the resultant of F_2' and F_3', and, since F_1' is the equilibrant of $-F_1$, it is likewise the equilibrant of F_2' and F_3'; in other words, the forces F_1', F_2', and F_3' (or their equivalents F_1, F_2, F_3) form a system of concurrent forces in equilibrium

Substantially the same explanations would apply in case there were external forces acting at A. These forces, with the forces F_1', F_2', and F_3', would form a system of balanced concurrent forces.

The problem of determining the forces acting on the members is thus reduced to the general problem of the equilibrium of concurrent forces. It is now necessary to ascertain what effect those forces have on the members of the frame—that is, what members are in tension and what members are in compression.

35. Character of the Stress in a Member.—Consider the condition of A_1A It has been explained that this member is held in equilibrium by the two forces F_1 and $-F_1$; this pair of forces constitutes the stress in the member, and the magnitude of either force is a measure of the stress (see *Fundamental Principles of Mechanics*). It will be observed that the tendency of the forces F_1 and $-F_1$ is to stretch the member, and that, therefore, the member is in tension. The stress in A_1A is, therefore, a tension of F_1—pounds, tons, etc., as the case may be. Similarly, the stress in A_2A is a compression equal to F_2 (by which is meant that each of the two forces constituting the stress has a magnitude equal to F_2), and the stress in A_3A is a tension equal to F_3.

36. In determining the character of the stress in a member, however, it is not necessary to consider the two forces acting at the extremities of the member. Referring again to the joint A, where the forces F_1', F_2', and F_3' (or their equals F_1, F_2, F_3) form a balanced system, it is seen that F_1' and F_3' have their arrowheads pointing away from the joint; these forces may be said to be *pulling* at the joint

in the direction of the members A_1A and A_2A respectively, and these two members are in tension. The force F_3', on the contrary, when supposed, as it should be, to be acting on the joint along A_3A, as shown by the dotted vector F_3'', *pushes* against the joint in the direction of the member A_3A; hence, this member is in compression.

37. Summing up *The force acting along a member measures tension or compression according as, when supposed to act on the joint through the member, it pulls at or presses on the joint.*

It has been seen, for instance, that F_3', when supposed to be acting on the joint through the member A_3A (as shown by F_3'') presses on that joint, and that A_3A is in compression.

38. The forces F_1', F_2', F_3' (which, for shortness, are called the stresses in the members) have been represented by vectors having their common origin at the joint. This is convenient in complicated problems in which the method of resolutes is used But, in almost all cases, the vectors may be drawn along the lines representing the members, and then the arrowhead on any force shows at once whether the force is (that is, measures) a pull or a thrust.

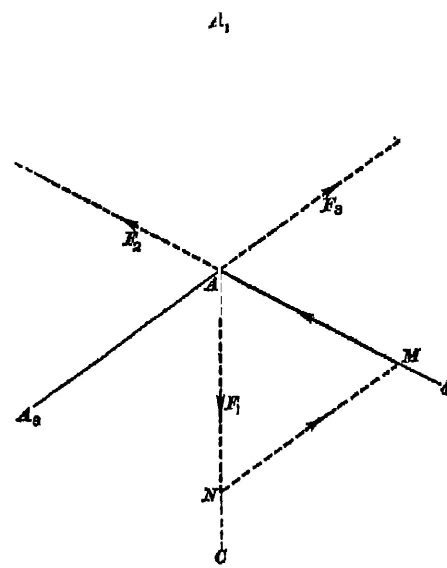


FIG. 20

39. Necessary Data For Determining Stresses. In determining the stresses in the members meeting at a joint, the external forces or the stresses in some of the other members must be known. Suppose, for instance, that the stress in the member A_1A

Fig. 20, is known to be a compression equal to F_1, represented by the vector AN, and that it is required to find the stresses F_2 and F_3 in the members A_2A and A_3A. Since the forces F_1, F_2, and F_3 are in equilibrium, a vector triangle can be constructed with the vectors representing them. Through N draw NM parallel to A_3A, meeting A_2A at M, and mark the arrowheads on NM and MA so that the vectors AN, NM, and MA will be in cyclic order. Then, $NM = F_3$; $MA = F_2$. As F_2 presses on the joint A, the member A_2A is in compression. Similarly, F_3, although given in direction by the vector NM, must be supposed to be acting along A_3A, and it is seen that, when thus applied, it presses on the joint A; therefore, A_3A also is in compression.

40. Opposite Arrowheads on a Member.—By referring to Fig. 21, it will be seen that the member connecting the two joints M and O has two arrowheads pointing in opposite directions. The student should not fall into the error of thinking that each arrowhead indicates the direction in which the part of the member on which it is marked acts on the other part. Thus, if the member is cut by a plane XY, the arrowhead F does not indicate the direction in which the part OZ acts on the part MZ; if that were the case, the member would be in compression, whereas it is in tension. The arrowhead F indicates the direction in which MO acts on O, and, as this action is transmitted from MZ to OZ, the arrowhead F really indicates the direction in which MZ acts on OZ; which shows that MZ pulls on OZ, and that, therefore, OZ is in tension. Likewise, the arrowhead F' indicates the direction in which the member acts on the joint M, or the direction in which any part of the member, as OZ, acts on the part MZ below it. The forces acting in

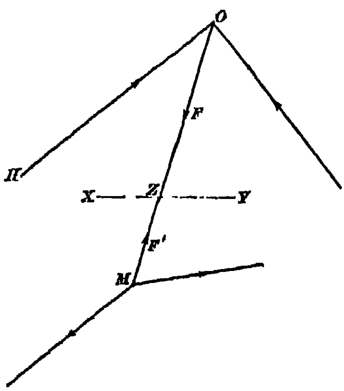


FIG 21

the direction of F and F' are, of course, equal, as each measures the stress in the member MO; either may be considered as the action, and the other as the equal and opposite reaction.

EXAMPLE.—A truss consisting of three horizontal members PC, CQ, and BD, Fig 22, and four equal inclined members PD, DC, CB,

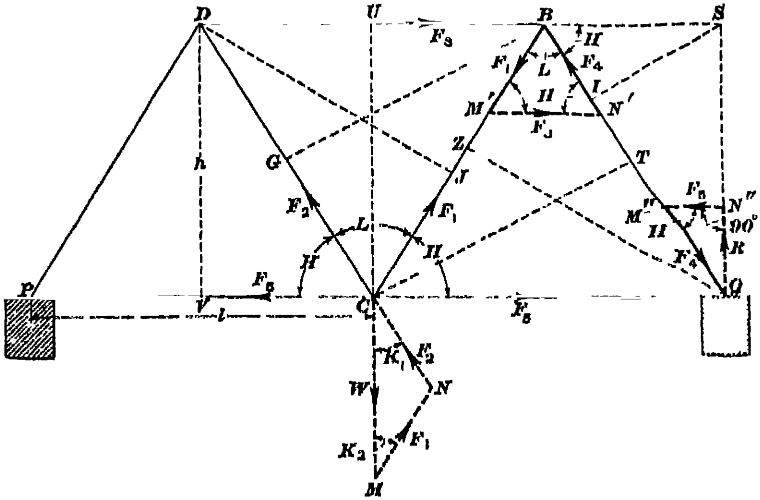


FIG 22

and BQ, carries a weight of W pounds hung from the center C. The truss rests on two piers P and Q; its length, between the joints at P and Q, is $2l$, and its height is h. Neglecting the weight of the structure, it is required to find the stress in each member and the reactions at the supports.

SOLUTION BY TRIANGLE OF FORCES.—On account of the symmetry of the figure, the angles PCD and QCB are equal; they are both denoted by H. Also, DC is parallel to BQ, and $DCB = CBQ = L$, say. The angles H and L, however, are not supposed to be known and have to be expressed in terms of l and h. The figure gives,

$$\sin H = \frac{DV}{CD} = \frac{h}{\sqrt{h^2 + \left(\frac{l}{2}\right)^2}} \qquad (a)$$

$$\tan H = \frac{DV}{CV} = \frac{2h}{l}$$

$$\cot H = \frac{1}{\tan H} = \frac{l}{2h} \qquad (b)$$

$$L + 2H = 180°$$

$$\sin L = \sin 2H = 2 \sin H \cos H \qquad (c)$$

Of the forces that balance at C, it is evident that, owing to the symmetry of the truss, the forces along CQ and CP are equal and opposite, and form, therefore, by themselves a balanced system. Hence, the weight W must be balanced by the forces F_1 and F_2 acting along the members CD and CB. The stresses in these two members are evidently pulls, and it is obvious that $F_1 = F_2$

In the triangle CMN, the vector CM represents the weight W, MN is parallel to CB and CN is in the prolongation of DC Therefore, $MN = F_1$ and $NC = F_2$. Since $F_1 = F_2$, then, also, $K_1 = K_2$

Now, $\quad K_1 = 180° - (90° + H) = 90° - H$

$$N = 180° - (K_1 + K_2) = 180° - 2K_1 = 180° - 2(90° - H) = 2H$$

Then, $$F_1 = F_2 = \frac{W}{\sin N}\sin K_1 = \frac{W}{\sin 2H}\sin(90° - H)$$

$$= \frac{W}{2\sin H\cos H}\cos H = \frac{W}{2\sin H}$$

Coming now to the joint B, it is known that the force F_1 acts on it along the member CB, hence, the two forces F_3 and F_4 acting along DB and QB can be determined Take $BM' = F_1$ and draw $M'N'$ parallel to DB, meeting BQ at N' Then, $M'N'$ represents the force F_3 acting along DB, and $N'B$ the force F_4 acting along QB The stresses in these members are evidently thrusts (Art 37) As $M'N'$ is parallel to CQ, the angles $BM'N'$ and $BN'M'$ are both equal to BCQ, or H. The triangle $BM'N'$ being isosceles, we have $F_4 = F_1$.

For F_3, we have,

$$F_3 = \frac{F_1}{\sin H}\sin L$$

Putting $F_1 = \frac{W}{2\sin H}$, as found above, and $\sin L = 2\sin H\cos H$,

$$F_3 = \frac{W}{2\sin^2 H} \times 2\sin H\cos H = W\cot H = \frac{Wl}{2h}$$

The forces acting at Q are the thrust F_4 of the member BQ, the horizontal force F_5 along CQ, and the reaction of the pier, which, as will be shown hereafter, acts vertically upwards Take QM'' to represent F_4 (notice that it is not necessary to measure the distance from Q in the direction of the arrowhead), draw $M''N''$ horizontal and QN'' vertical, and, starting with F_4, whose direction is known, mark the arrowheads on the sides of the triangle $M''QN''$ in cyclic order Then, QN'' will represent the reaction R and $N''M''$ the force F_5 in QC. The latter force acts in the direction QC, thus showing the stress in QC to be a pull. The triangle $M''QN''$ gives

$$F_5 = F_4\cos H = F_1\cos H = \frac{W\cos H}{2\sin H} = \frac{W}{2}\cot H = \frac{Wl}{4h}$$

For the reaction,

$$R = F_5\tan H = \frac{W}{2}\cot H\tan H = \frac{W}{2}$$

The reaction being equal and opposed to the pressure exerted by the truss on the pier, or to the part of the load transmitted by the truss to the pier, it is seen that one half of the load is transmitted to Q, and, as the truss is symmetrical, the other half is transmitted to the other pier P. This subject will be more fully explained in connection with the theory of parallel forces. By this theory, the value of the reaction can be determined first and the calculation begun at the joint through which the reaction acts, this is the method used in practice.

On account of the symmetry of the truss, the stress in DP is obviously the same as the stress in BQ.

SOLUTION BY MOMENTS.—The solution of this problem by the method of moments is as follows:

For joint C, moments are taken about a point on the line of action of one of the unknown forces F_1 or F_2. The equal and opposite forces F_4 need not be taken into account. If the joint B is taken as the origin of moments, the lever arm of W is $BU = \frac{1}{2} l$, and that of F_2 is

$$BG = BC \sin L = DC \sin L = \frac{CV}{\cos H} \sin L = \frac{l \sin L}{2 \cos H}$$

The moment of W is negative, and that of F_2 is positive (Art. **18**). Therefore (Art. **24**),

$$-W \times BU + F_2 \times BG = 0,$$

or, writing the values of BU and BG just found,

$$\frac{Wl}{2} - \frac{F_2 l \sin L}{2 \cos H} = 0,$$

whence,
$$F_2 = \frac{W \cos H}{\sin L}$$

This was the value found for F_1 (which is equal to F_2) by using the triangle of forces, and may be transformed in the same manner.

Passing now to joint B, F_4 is determined by taking moments about D, in the line of action of F_3. The lever arms of F_1 is DJ, and the lever arm of F_4 is the perpendicular from D on QB produced, but this perpendicular is equal to BG, because QB and CD are parallel.

Therefore,
$$F_1 \times DJ - F_4 \times BG = 0,$$

whence,
$$F_4 = F_1 \times \frac{DJ}{BG} = F_1$$

because the triangle BCD is isosceles, and $DJ = BG$.

To find F_3, take moments about Q, in the line of F_4. The lever arm of F_3 is $QS = h$, that of F_1 is $QZ = l \sin H$. Then,

$$F_3 h - F_1 l \sin H = 0$$

$$F_3 = \frac{F_1 l \sin H}{h};$$

or, because $F_1 = \frac{W}{2 \sin H}$ (see the solution by the triangle of forces given above),

$$F_3 = \frac{Wl}{2h}$$

Passing now to joint Q, to find F_5, take moments about S in the line of R. The lever arm of F_4 is $S I = B S \sin H = \frac{l}{2} \sin H$, that of F_5 is $Q S = h$. Then,

$$F_5 h - F_4 \times \frac{l \sin H}{2} = 0,$$

whence,

$$F_5 = \frac{F_4 l \sin H}{2h} = \frac{F_1 l \sin H}{2h},$$

but (see above) $\frac{F_1 l \sin H}{h} = F_3$, therefore,

$$F_5 = \frac{F_3}{2} = \frac{W l}{4h}$$

To find R, take moments about C, in the line of F_5. The lever arm of R is $Q C = l$, that of F_4 is $C T = Q Z = l \sin H$.

Then, $F_4 l \sin H - R l = 0$, whence,

$$R = F_4 \sin H = F_1 \sin H = \frac{W}{2 \sin H} \times \sin H = \frac{W}{2}$$

In practice, it is not necessary to draw the perpendiculars $C U$, $B G$, $C T$, etc., as their values can be at once written down by means of the fundamental trigonometric relations among the elements of a right triangle.

EXAMPLES FOR PRACTICE

1. A weight of 2 tons is suspended from a derrick, Fig. 23; the length of the boom $A B$ is 40 feet; the guy rope $C B$ is fastened at a point C, 30 feet from A, and the boom is 10 feet out of vertical. Find

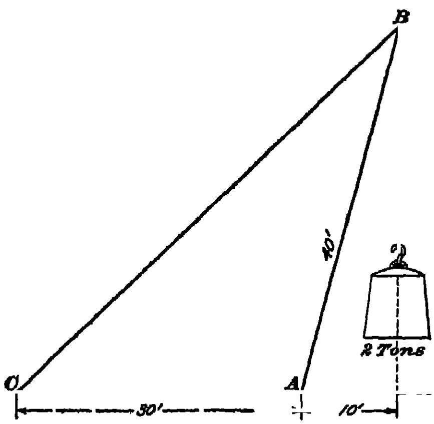


FIG 23

the tension F_1 in the guy rope and the thrust F_2 in the boom.

Ans. $\begin{cases} F_1 = .95840 \text{ tons} \\ F_2 = 2\ 7542 \text{ tons} \end{cases}$

2 A trapezoidal frame having the dimensions shown in Fig 24 rests on two piers and carries two weights of 5 tons each at the joints A and B Find the stresses F_1, F_2, and F_3 in the members $A B$,

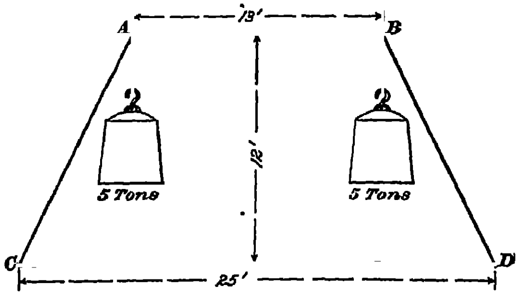


FIG 24

$A C$, and $C D$, respectively, and the reaction R at either pier The members $A B$ and $C D$ are horizontal, and the inclinations of $A C$ and $B D$ to $C D$ are equal.

$$\text{Ans}\begin{cases} F_1 = 2.5 \text{ tons, thrust} \\ F_2 = 5.5902 \text{ tons, thrust} \\ F_3 = 2.5 \text{ tons, pull} \\ R = 5 \text{ tons} \end{cases}$$

PARALLEL FORCES

COPLANAR FORCES

TWO FORCES

41. Resultant of Two Parallel Forces Having the Same Direction.—Let two parallel forces F_1 and F_2, Fig. 25, act on a rigid body $A B C D$. The lines of action of the two forces are $K_1 L_1$ and $K_2 L_2$, respectively, and the points of application E_1 and E_2 are any two points on those lines (see Art 10). Draw $E_1 E_2$. The effect of the forces will not be changed if any two equal and opposite forces $E_1 T_1$ and $E_2 T_2$, acting along the line $E_1 E_2$, are introduced, for these two forces will evidently balance each other. The two forces F_1 and F_2 may, therefore, be replaced by the four forces F_1 and $E_1 T_1$, F_2 and $E_2 T_2$. The resultant of F_1 and $E_1 T_1$ is $E_1 S_1$, the diagonal of the parallelogram $E_1 N_1 S_1 T_1$. Similarly, the resultant of F_2 and $E_2 T_2$ is $E_2 S_2$. The lines of action

of $E_1 S_1$ and $E_2 S_2$ meet at a point E'. By the principle of the transferability of the point of application (Art. **10**), the points of application of $E_1 S_1$ and $E_2 S_2$ may be transferred to E', provided that the point E' is supposed to be rigidly connected to the body $ABCD$. In this new position, the forces are represented by $E' S_1'$ and $E' S_2'$. The force $E' S_1'$ may now be resolved again into its components $E' T_1'$, equal and parallel to $E_1 T_1$, and $E' N_1'$, equal and parallel to F_1;

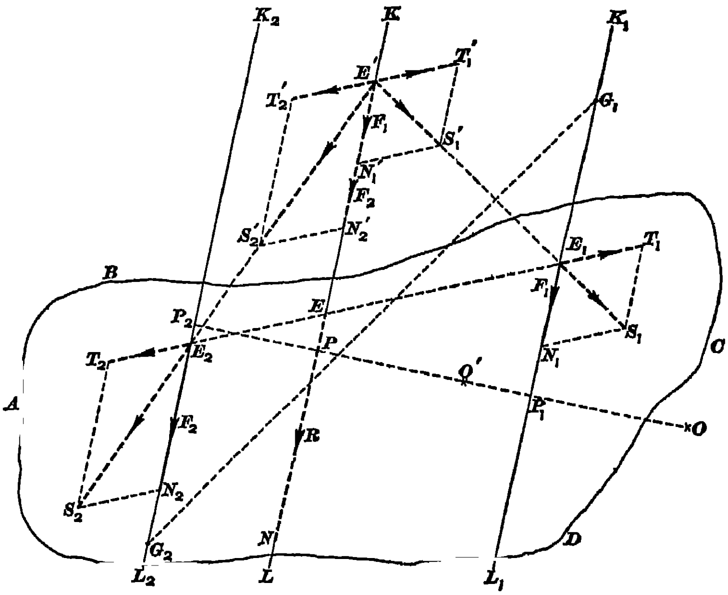


FIG 25

and $E' S_2'$ into its components $E' T_2'$, equal and parallel to $E_2 T_2$, and $E' N_2'$, equal and parallel to F_2. As $E' T_1'$ and $E' T_2'$ balance each other, only two forces, acting along the line KL, now remain, whose resultant R is equal to $F_1 + F_2$.

The point of application of this force may be taken anywhere, as at E, on the line KL.

In the similar triangles $E E' E_1$ and $N_1' E' S_1'$ we have

$$\frac{E E_1}{N_1' S_1'} = \frac{E' E}{E' N_1'}; \text{ whence}$$

$$E E_1 \times E' N_1' = E' E \times N_1' S_1' \qquad (a)$$

In the same manner, we get, from the triangles $EE'E_2$ and $N_2'E'S_2'$,

$$EE_2 \times E'N_2' = E'E \times N_2'S_2'$$

or, because $N_2'S_2' = E'T_2' = E'T_1' = N_1'S_1'$,

$$EE_2 \times E'N_2' = E'E \times N_1'S_1'$$

This, compared with (a), gives

$$EE_1 \times E'N_1' = EE_2 \times E'N_2',$$

that is,

$$EE_1 \times F_1 = EE_2 \times F_2,\ F_1 \times EE_1 = F_2 \times EE_2 \qquad (b)$$

whence,

$$\frac{EE_1}{EE_2} = \frac{F_2}{F_1} \qquad (c)$$

From this we obtain, according to the laws of proportion (see *Geometry*),

$$\frac{EE_1 + EE_2}{EE_2} = \frac{F_1 + F_2}{F_1};$$

that is,

$$\frac{E_1E_2}{EE_2} = \frac{R}{F_1}, \text{ and } F_1 \times E_1E_2 = R \times EE_2 \qquad (d)$$

Likewise,

$$\frac{E_1E_2}{EE_1} = \frac{R}{F_2}, \text{ and } F_2 \times E_1E_2 = R \times EE_1 \qquad (e)$$

The preceding results may be stated as follows

1. *The resultant of two parallel forces having the same direction is equal to their sum, its line of action is parallel to the lines of action of the two forces, and its direction is the same as the common direction of the two forces.*

2. *The line of action of the resultant is so situated that it divides any line intercepted between the components into two segments inversely proportional to those components* [equation (c)].

3. *The resultant and the components are so related and situated that, if the points of application of the three are taken on any straight line* (as E_1E_2, Fig. 25) *intersecting their lines of action, the products of any two of the three forces by the distances of their respective points of application from the point of application of the third force are equal* [equations (b), (d), (e)].

The last statement applies to distances taken on any line between the lines of action of the two forces; for it will be remembered that E_1 and E_2 were taken arbitrarily on K_1L_1

and $K_2 L_2$. Any other points, as G_1 and G_2, might have been taken, and the same reasoning would have led to the same conclusions

Let $E_1 E_2 = l$, $E E_1 = l_1$, $E E_2 = l_2$. Then, equations (b), (d), and (e) may be written

$$\left.\begin{aligned} F_1 l_1 &= F_2 l_2 \\ F_1 l &= R l_2 \\ F_2 l &= R l_1 \end{aligned}\right\} \qquad (f)$$

42. Equilibrium of Three Parallel Forces.—By reversing the direction of R, Fig 25, we have the equilibrant Q of the forces F_1 and F_2 Arithmetically, the value of Q is $F_1 + F_2$. But, if we consider forces acting in one direction as positive and those acting in the opposite direction as negative, we have

$$Q = -R = -F_1 - F_2;$$

whence, $$Q + F_1 + F_2 = 0$$

This, with any one of equations (f) of Art. 41, gives the conditions of equilibrium of three parallel forces. The equilibrant must always lie between the other two forces.

43. Theorem of Moments.—Let O, Fig. 25, be any point on the plane of the forces, and draw $O P_2$ perpendicular to their lines of action. Let M_1, M_2, M_r be the moments of F_1, F_2, and R about O Then,

$$M_1 = F_1 \times O P_1 \qquad (a)$$

$$M_2 = F_2 \times (O P_1 + P_1 P_2) \qquad (b)$$

$$M_r = R \times (O P_1 + P_1 P)$$

Adding (a) and (b),

$$\begin{aligned} M_1 + M_2 &= (F_1 + F_2) \times O P_1 + F_2 \times P_1 P_2 \\ &= R \times O P_1 + F_2 \times P_1 P_2 \end{aligned}$$

or, because $F_2 \times P_1 P_2 = R \times P_1 P$ (see Art. 41),

$$M_1 + M_2 = R \times O P_1 + R \times P_1 P = R \times (O P_1 + P_1 P) = M_r$$

Therefore, *the moment of the resultant of two parallel forces about any point in their plane equals the algebraic sum of the moments of the components.*

The student may take the point O' and verify this principle, paying due attention to the signs of the moments, according to the convention explained in Art. 18.

If, instead of the resultant, the equilibrant Q is taken, its moment M_q is equal to $-M_r$, and

$$M_q = -M_r = -M_1 - M_2$$

whence,

$$M_q + M_1 + M_2 = 0$$

44. Two Parallel Forces Acting in Opposite Directions.—When the two component forces act in opposite directions, as F_1 and F_2, Fig. 26, the resultant is found as follows. If a force $Q = F_1 - F_2$ is applied at a point E in $E_2 E_1$ produced, such that $F_1 l = Q l_2$, the three forces F_1, F_2, and Q will be in equilibrium, according to the principles stated in Arts. **42** and **43.** Since Q balances F_1 and F_2, it must be equal and opposite to their resultant R. Therefore, in this case, the resultant is equal to the difference of the components, and the line $E_2 E_1$ produced is divided by the three forces F_1, F_2, and R in the manner indicated by equations (f) of Art. **41.** The law of moments applies in this case also, but due attention must be paid to signs

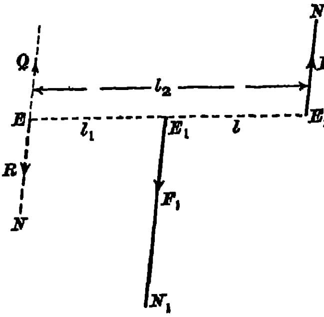


FIG 26

45. The principle of moments just stated for the case of two parallel forces and their resultant (or of three balanced parallel forces) is known as the **principle of the lever,** or the **law of the lever,** as it was first discovered by Archimedes in the determination of the conditions of equilibrium of a lever.

46. Definition of a Couple.—There is an apparent exception to the foregoing conclusions that must be particularly noticed. If $F_1 = F_2$, Fig. 26, the formulas would give $R = F_1 - F_2 = 0$, and, from equations (f) of Art. **41,**

$$l_2 = \frac{F_1 l}{R} = \frac{F_1 l}{0}, \text{ and } l_1 = \frac{F_2 l}{R} = \frac{F_2 l}{0}$$

So, although the resultant is 0, the distance of its point of application from the points of application of the other two forces cannot be determined, since the fractions $\frac{F_1 l}{0}$ and $\frac{F_2 l}{0}$

do not represent any numbers. This simply means that, in the case here considered, it is impossible to replace the two given forces by a single force.

A system of two equal parallel forces acting in opposite directions but not in the same line, is called a **couple**.

The theory of couples is very important, and will be more fully treated further on. But here it may be stated, from what has just been explained, that *a couple cannot be either replaced or balanced by a single force.*

47. Resolution of a Force Into Two Parallel Components.—Equations (f) of Art. 41 afford a means to resolve any force into two parallel components passing through any two given points situated on a line intersecting the line of action of the given force. For, if R, l_1, and l_2 are given, we have:

$$R\,l_1 = F_2\,l = F_2\,(l_1 + l_2);$$

whence,
$$F_2 = \frac{R\,l_1}{l_1 + l_2}$$

Similarly,
$$F_1 = \frac{R\,l_2}{l_1 + l_2}$$

If R and one of the distances, as l_1, from the line of action of R to the line of action of one of the components are given, and also the magnitude F_1 of this component, the other component is found from the relation

$$F_2 = R - F_1$$

Its distance l_2 from the line of action of R is determined as follows:

$$F_2\,l_2 = F_1\,l_1;$$

whence, $l_2 = \dfrac{F_1}{F_2}l_1 = \dfrac{F_1}{R - F_1}l_1$

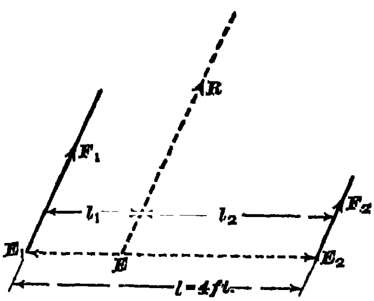


FIG. 27

The distances l_1 and l_2 may be either perpendicular or oblique.

EXAMPLE 1.—It is required to find the magnitude and line of action of the resultant R of the parallel forces F_1(= 500 pounds) and F_2(= 250 pounds), the distance l being 4 feet, as shown in Fig. 27.

SOLUTION —According to statement 1 of Art **41**, the resultant is equal to the sum of F_1 and F_2, that is, $R = 500 + 250 = 750$ lb Ans

The distance l_1 from the line of action of F_1 to the line of action of R, along the line l, is given by equations (f) of Art **41**:

$$F_2 l = R\,l_1, \text{ from which, } l_1 = \frac{F_2 l}{R}$$

From the above, $F_2 = 250$, $l = 4$, and $R = 750$, therefore,

$$l_1 = \frac{250 \times 4}{750} = 1\tfrac{1}{3} \text{ ft} \quad \text{Ans}$$

$$l_2 = \frac{F_1 l}{R} = \frac{500 \times 4}{750} = 2\tfrac{2}{3} \text{ ft} \quad \text{Ans}$$

EXAMPLE 2 —In Fig 28, the forces $F_1 (= 200$ pounds) and $F_2 (= 50$ pounds) act in opposite directions; it is required to find the resultant R when the distance l is 6 feet.

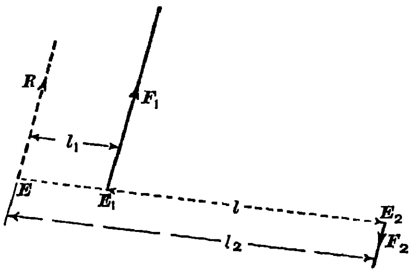


FIG 28

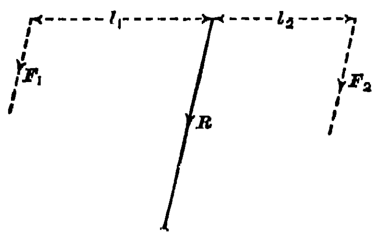


FIG 29

SOLUTION —The resultant is equal to the difference between the forces (Art. **44**); that is, $R = F_1 - F_2 = 200$ lb. $- 50$ lb $= 150$ lb. Ans.

From equations (f) of Art **41**,

$$\left.\begin{aligned} l_2 &= \frac{F_1 l}{R} = \frac{200 \times 6}{150} = 8 \text{ ft} \\ l_1 &= \frac{F_2 l}{R} = \frac{50 \times 6}{150} = 2 \text{ ft.} \end{aligned}\right\}\text{Ans.}$$

EXAMPLE 3 —In Fig 29, resolve the force $R (= 450$ pounds) into two parallel forces F_1 and F_2, when $l_1 = 10$ feet and $l_2 = 8$ feet

SOLUTION.—From equations (f) of Art **41**,

$$F_1 = \frac{R\,l_2}{l}, \text{ and } F_2 = \frac{R\,l_1}{l}$$

In this example, $R = 450$ lb, $l_1 = 10$ ft., $l_2 = 8$ ft., and $l = l_1 + l_2 = 10 + 8 = 18$ ft. Therefore,

$$\left.\begin{aligned} F_1 &= \frac{450 \times 8}{18} = 200 \text{ lb.} \\ F_2 &= \frac{450 \times 10}{18} = 250 \text{ lb.} \end{aligned}\right\}\text{Ans.}$$

ANY NUMBER OF FORCES

48. Magnitude of the Resultant.—Let F_1, F_2, F_3, F_4, Fig 30, be parallel coplanar forces acting through points A_1, A_2, etc of a rigid body, their lines of action being, respectively, $K_1 L_1$, $K_2 L_2$, etc. By the principle of the transferability of the point of application, each force may be sup-

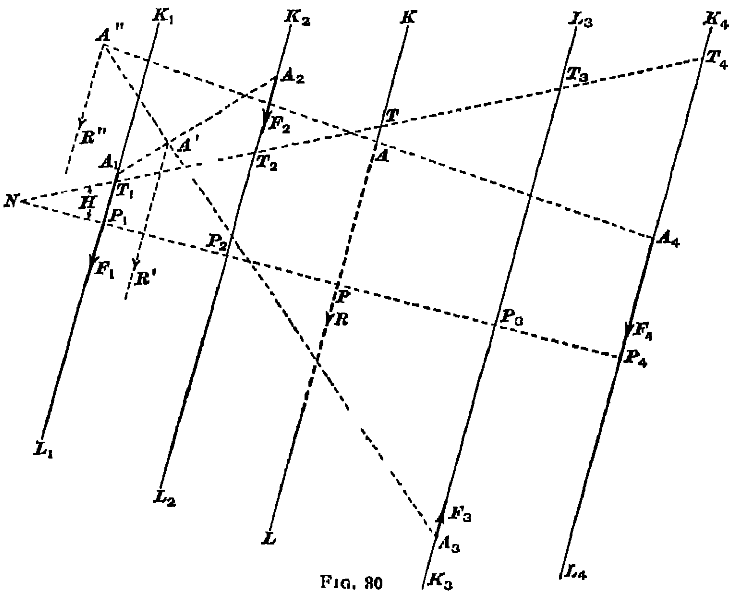


Fig. 30

posed to be applied at any point on its line of action; but, for reasons presently to be explained, it will be assumed that the points of application are the fixed points A_1, A_2, etc.

In the first place, *the resultant R is equal to the algebraic sum of the components, and its line of action is parallel to the lines of action of the components*

For, according to Arts. **41** and **44**, the resultant R' of F_1 and F_2 is parallel to F_1 and F_2, and equal to $F_1 + F_2$; the resultant R'' of R' and F_3 is parallel to R' and F_3, and equal to $R' + F_3 = F_1 + F_2 + F_3$ (the negative sign being implied); the resultant R of R'' and F_4 is parallel to R'' and F_4, and equal to $R'' + F_4 = F_1 + F_2 + F_3 + F_4$.

The same reasoning applies to any number of forces; so that, if their algebraic sum is denoted by ΣF, then $R = \Sigma F$.

49. Line of Action of the Resultant.—To locate the line of action KL of the resultant, let N be any point in the plane of the forces, and NP_4 a perpendicular to the common direction of the forces, intersecting K_1L_1 at P_1, K_2L_2 at P_2, etc.; and KL, the still unknown line of action of the resultant, at P Let $NP_1 = p_1$, $NP_2 = p_2$, etc., and $NP = p_r$

If moments are taken about N, the following equations obtain (Art 43).

$$\begin{aligned}
\text{Moment of } R' &= F_1p_1 + F_2p_2 \\
\text{Moment of } R'' &= \text{moment of } R' + F_3p_3 \\
&= F_1p_1 + F_2p_2 + F_3p_3 \\
\text{Moment of } R &= Rp = \text{moment of } R'' + F_4p_4 \\
&= F_1p_1 + F_2p_2 + F_3p_3 + F_4p_4
\end{aligned}$$

In general, if the algebraic sum of the moments of any number of coplanar parallel forces is denoted by ΣFp, and the lever arm of the resultant R by p_r, then

$$Rp_r = \Sigma Fp;$$

whence, $$p_r = \frac{\Sigma Fp}{R} = \frac{\Sigma Fp}{\Sigma F} \qquad (1)$$

This locates the line of action of R with respect to the point N.

If, instead of the perpendicular distances p_1, p_2, etc. of the lines of action of the forces from N, the oblique distances $NT_1 = a_1$, $NT_2 = a_2$, etc., along any line NT_4 (as when the forces are applied at several points of the same straight line), are given, the following equations are obtained:

$$\begin{aligned}
p_1 &= NP_1 = NT_1 \cos H = a_1 \cos H \\
p_2 &= NP_2 = NT_2 \cos H = a_2 \cos H, \text{ etc.} \\
p_r &= NP = NT \cos H = a_r \cos H
\end{aligned}$$

Substituting these values of p_1, p_2, etc. in the equation of moments,

$$Ra_r \cos H = F_1a_1 \cos H + F_2a_2 \cos H + \ldots \text{ etc.};$$

whence, $$a_r = \frac{F_1a_1 + F_2a_2 + \ldots}{R} = \frac{\Sigma Fa}{\Sigma F} \qquad (2)$$

If N is taken on the line of action of one of the forces, that force is eliminated from formulas **1** and **2**, since in this case both its p and its a are equal to zero Thus, for two forces, the preceding equations become identical with equations (f) of Art. **41.**

It should be carefully borne in mind that, in using the inclined distances a_1, a_2, etc., the same rules for signs are to be observed as for the perpendicular distances p_1, p_2, etc. Thus, $F_1 a_1$ is to be taken as positive and $F_2 a_2$ as negative.

50. Conditions of Equilibrium.—When the forces are in equilibrium, we must have $\Sigma F = 0$, and $\Sigma(Fp) = 0$; that is, *the algebraic sum of the forces must be zero, and the algebraic sum of their moments about any point in their plane must be zero*

Both of these conditions are necessary for equilibrium For, in the first place, it is evident that if ΣF is not zero, there is a resultant, and no equilibrium is possible. In the second place, if the resultant of all the forces acting in one direction is equal to the resultant of all the forces acting in the opposite direction, the condition $\Sigma F = 0$ obtains, but, if those two resultants have not the same line of action [in which case $\Sigma(Fp)$ is not zero], they form a couple (Art. **46**), and the system cannot be in equilibrium

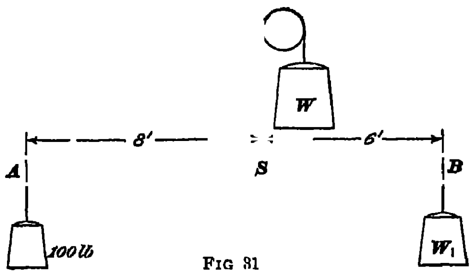


FIG 31

It should be remembered that the arithmetical meaning of the expression $\Sigma F = 0$ is that *the arithmetical sum of all the forces acting in one direction is equal to the arithmetical sum of all the forces acting in the opposite direction.*

EXAMPLE 1 —A weight of 100 pounds is hung from the extremity A of a straight lever $A B$, Fig. 31. The lever is suspended by a rope passed around a smooth peg and carrying a weight W; and from the extremity B is hung another weight W_1. The dimensions being as shown, it is required to find the magnitudes of W_1 and W, that the lever may remain in equilibrium. (The weight of the lever is neglected.)

SOLUTION.—Since W is the equilibrant of W_1 and 100 lb., the sum of the moments of the three forces about any point in their plane must be zero. If moments are taken about S, which is in the line of action of W (for W acts upwards at S), the force W will be eliminated, its moment about S being zero. The moment of W_1 being positive, and that of the 100-lb. weight negative, we have,

$$6\,W_1 - 100 \times 8 = 0,$$

whence, $$W_1 = \frac{800}{6} = 133.33 \text{ lb.} \quad \text{Ans.}$$

To find W, we have,

$$100 + W_1 - W = 0$$

$$W = 100 + W_1 = 100 + 133.33 = 233.33 \text{ lb.} \quad \text{Ans.}$$

Notice that it is not necessary that the lever should be horizontal. From the mathematical conditions of equilibrium (Art. **50**), it follows that the three weights given above will hold the lever in equilibrium in any position.

EXAMPLE 2.—A rigid bar AB, Fig. 32, rests on two supports S_1 and S_2,

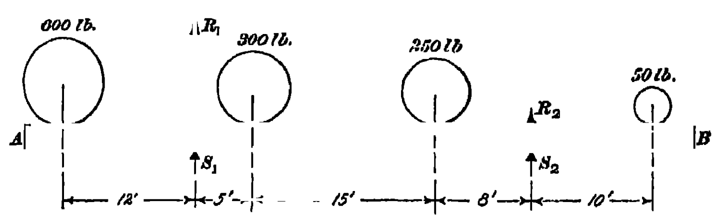


FIG. 32

and carries weights of 600, 300, 250, and 50 pounds placed as shown. The weight of the bar being neglected, it is required to find the reactions R_1 and R_2 at S_1 and S_2, respectively.

SOLUTION.—Here we have a system of balanced parallel forces, consisting of the given weights, which act downwards, and the two reactions, which act upwards. Taking moments about S_2, and observing the rule of signs, we have,

$$-600 \times 40 + R_1 \times 28 - 300 \times 23 - 250 \times 8 + 50 \times 10 = 0;$$

whence, $$R_1 = \frac{32,400}{28} = 1,157.1 \text{ lb.}$$

To find R_2, we have:

$$R_1 + R_2 = 600 + 300 + 250 + 50 = 1,200,$$

whence, $$R_2 = 1,200 - R_1 = 42.9 \text{ lb.}$$

In order to check these results, the value of R_2 may be found by taking moments about S_1. The moments of the weights at the right

of S_1 are positive, the moment of R_2 and the moment of the weight at the left of S_1 are negative. Therefore,

$$-600 \times 12 + 300 \times 5 + 250 \times 20 - R_2 \times 28 + 50 \times 38 = 0,$$

whence, $R_2 = \frac{1,200}{28} = 42.9$ lb, approximate to tenths.

EXAMPLES FOR PRACTICE

NOTE —In these examples the weights of bars and ropes are neglected

1 A straight bar is supported at two points S_1 and S_2 25 feet apart, and carries five loads between the supports as follows 100 pounds, placed 3 feet from S_1, 150 pounds, placed 8 feet from S_1, 200 pounds, placed in the middle of the bar, 300 pounds, placed 15 feet from S_1, and 450 pounds, placed 20 feet from S_1 Find the reactions R_1 and R_2 at S_1 and S_2

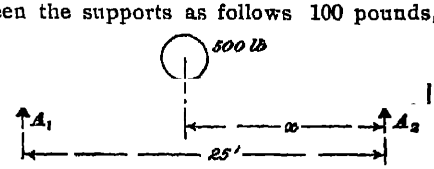


FIG 33

Ans. $R_1 = 500$ lb.; $R_2 = 700$ lb

2. A man carries a weight of 30 pounds hung from a stick resting on his shoulder, the distance from the weight to the shoulder is 2 5 feet. Find the pressure P on the shoulder· (a) when the man holds the stick at a distance of 1 foot from the shoulder, (b) when he holds the stick at a distance of 1.5 feet from the shoulder.

$$\text{Ans.} \begin{cases} (a) & P = 105 \text{ lb} \\ (b) & P = 80 \text{ lb} \end{cases}$$

3 A rigid bar, supported on two piers A_1 and A_2, Fig 33, is to carry a weight of 500 pounds. The pier A_1 will crush under a pressure exceeding 275 pounds, but it is desired to have the weight as far from A_2 as possible Find the maximum distance x at which the weight can be placed from A_1

Ans $x = 13$ 75 ft

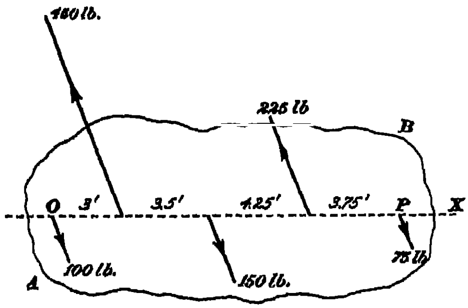


FIG 34

4 Five coplanar parallel forces act on a body AB, Fig 34; magnitudes and direction of forces, and distances along OX, are as shown Find the magnitude and direction of their resultant R, and the distance a of its line of action from O, measured along the line OX.

Ans. $R = 350$ lb., acting upwards; $a = 4$ 875 ft to the right of O.

NOTE —First take moments about O, using the inclined distances instead of the lever arms; then, as a check, take moments about P.

5 In example 4, find two forces F_1 and F_2 parallel to the given forces and equivalent to them, F_1 to act at a distance of 10 feet to the right of O, and F_2 at a distance of 14 feet to the right of O

Ans $\begin{cases} F_1 = 798\ 44 \text{ lb, acting upwards} \\ F_2 = 448\ 44 \text{ lb, acting downwards} \end{cases}$

6 A bar $A_1 A_2$, Fig 35, is to carry two equal weights W, suspended as shown The bar is supported by two ropes $A_1 B_1$ and $A_2 B_2$, of which $A_1 B_1$ cannot be subjected to a tension greater than 1,000 pounds, and $A_2 B_2$ cannot be subjected to a tension greater than 600 pounds. What is the greatest value that W can have, and what are the tensions F_1 and F_2 in the two ropes when the bar carries its greatest load?

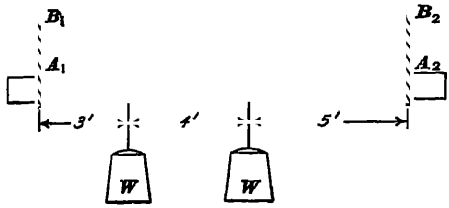


FIG 35

Ans $\begin{cases} W = 720 \text{ lb} \\ F_1 = 840 \text{ lb} \\ F_2 = 600 \text{ lb.} \end{cases}$

NOTE —Assume first $F_1 = 1{,}000$, and find W by taking moments about A_2; then, $F_2 = 2W - F_1$ As this is greater than 600, start by assuming $F_2 = 600$ pounds, and proceed to find W and F_1

NON-COPLANAR PARALLEL FORCES

51. Magnitude of the Resultant of Any Number of Parallel Forces. Let F_1, F_2, F_3, F_4, Fig. 36, be parallel forces acting on a body at the points A_1, A_2, etc. The forces may be either coplanar or not. In either case, they may be combined in pairs, as in Art. 48. Hence, the resultant is equal to the algebraic sum of the components.

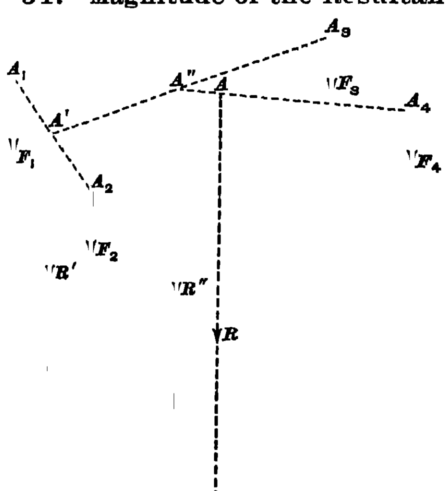


FIG 36

52. Center of Any System of Parallel Forces. According to the principles stated in Art. 41,

the resultant R' of F_1 and F_2, Fig. 36, passes through a point A' in the line A_1A_2, whose distance from A_1 is $A_1A_2 \times \frac{F_2}{R'} = A_1A_2 \times \frac{F_2}{F_1 + F_2}$. Since this distance does not depend on the inclination of the forces to A_1A_2, and since the fraction $\frac{F_2}{F_1 + F_2}$ remains the same when for F_1 and F_2 are substituted any forces proportional to the latter, such as nF_1 and nF_2 (n being any number, integral or fractional), it follows that, so long as the points A_1 and A_2 remain fixed and the forces remain parallel and their relative magnitudes and directions unchanged, the lines of action of the forces may be revolved about A_1 and A_2 and made parallel to any direction whatever in space, without changing the position of the point A', through which the resultant R' must constantly pass. The same principle may be proved of the point A'' through which the resultant R'' of R' and F_3 must pass, and of the point A, traversed by the total resultant R. It follows that, *for every system of parallel forces applied at—or, more properly, passing through—any points whatever, there is a fixed point through which the resultant must pass, whatever the (common) direction of the forces may be. Furthermore, that point is the same for all systems of parallel forces in which the relative magnitudes of the forces are the same, whatever their absolute magnitudes may be.*

Such a point, as A in Fig. 36, is called the **center** of the system of forces considered.

53. Center of Gravity Defined.—In the particular case in which the points of application A_1, A_2, etc. are the particles of a body, and the forces acting are the weights of those particles, the center of the system of parallel forces thus formed is called the **center of gravity** of the body. The weight of the body, which is the resultant of the weights of its particles, must therefore be treated as a force whose line of action passes through the center of gravity of the body. The methods for determining the position of the center of gravity will be given later.

54. Coordinates of Center of Parallel Forces.—When all the points of application of the forces lie in one plane, the center of the forces is easily found as follows:

Let the parallel forces F_1, F_2, F_3 act at the points A_1, A_2, A_3, Fig. 37, and let OX and OY be any two mutually perpendicular lines drawn in the plane containing A_1, A_2, A_3. These reference lines are called **coordinate axes**, and the distances of any point from them are called the **coordinates** of that point. Distances perpendicular to OY, or parallel to OX, are usually denoted by the letter x, sometimes with an accent or subscript; distances perpendicular to OX, or parallel to OY, are denoted by the letter y. Distances above OX and

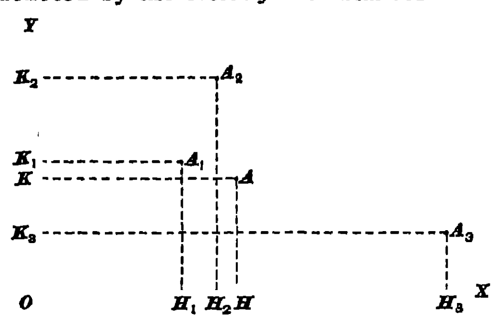


Fig 37

those to the right of OY are treated as positive; distances below OX, and those to the left of OY, as negative.

Let the coordinates of A_1 be $OH_1 = K_1A_1 = x_1$ and $OK_1 = H_1A_1 = y_1$. Similarly, let the coordinates of A_2 be x_2 and y_2; those of A_3 be x_3 and y_3; and those (still unknown) of the center A of the forces be x_c and y_c.

Since the position of A is independent of the common direction of the forces, the latter may be imagined as acting in the plane XOY in a direction parallel to OX. If, now, moments are taken about any point on OX, the lever arm of F_1 will evidently be equal to $A_1H_1 = y_1$, the lever arm of F_2 will be equal to y_2, that of A_3 equal to y_3, and that of R equal to y_c. Then, by formula **1** of Art. **49**,

$$y_c = \frac{F_1y_1 + F_2y_2 + F_3y_3}{F_1 + F_2 + F_3}$$

If, on the contrary, the forces are imagined as acting in a direction parallel to OY, and moments are taken about any point on OY, the following formula is obtained, by pursuing the same method as before.

$$x_c = \frac{F_1 x_1 + F_2 x_2 + F_3 x_3}{F_1 + F_2 + F_3}$$

The coordinates x_c and y_c locate the center A.

As the foregoing reasoning is evidently applicable to any number of parallel forces whose points of application all lie in one plane, the following general formulas can be at once written

$$x_c = \frac{\Sigma Fx}{\Sigma F} \qquad (1)$$

$$y_c = \frac{\Sigma Fy}{\Sigma F} \qquad (2)$$

55. Moment About a Line.—The product $F_1 y_1$ considered in the last article is called the moment of the force F_1 **about the line** OX, **or with respect to** OX. In general, when a force is considered as applied to a special point, the moment of the force about any line perpendicular to the line of action of (although not necessarily in the same plane with) the force is the product of the magnitude of the force and the perpendicular distance of the point of application of the force from that line.

ANALYTIC STATICS

(PART 2)

CENTER OF GRAVITY

DEFINITIONS AND GENERAL PROPERTIES

1. The **center of gravity** of a body has already been defined as the center of the parallel forces of gravity acting on the particles of the body, or the center of the weights of all the particles, each particle being taken as the point of application of its own weight (see *Analytic Statics*, Part 1). These forces are considered parallel because they are all directed toward the earth's center, whose distance from the surface is so great, compared with the dimensions of ordinary bodies, that the angle between the lines of action of the weights of any two particles of a body is practically zero. (Two terrestrial radii meeting the surface at two points distant 100 feet from each other make an angle at the center equal to about 1 second.)

The abbreviation c. g. will here be used to signify center of gravity. The c. g. of a body is called also **center of mass, center of inertia,** and **centroid.** These terms, however, will not be used here.

2. Immediate Consequences of the Definition. Three important consequences follow at once from the preceding definition, namely:

1. As stated in *Analytic Statics*, Part 1, *the weight of a body may be treated as a single force acting vertically through the c. g. of the body.* This is expressed by saying that the

weight of a body may be supposed **concentrated** at the c. g of the body

2 *The position of the c. g. of a homogeneous body (that is, a body whose substance is the same throughout, or whose particles have all the same weight) depends on the form of the body only, not on the material of which the body is made.* For the center of a system of parallel forces depends on the relative, not on the absolute, magnitudes of the forces, and on the relative positions of their points of application.

3 *If the c. g. of a body acted on by its own weight is supported, the body will remain in equilibrium, whatever its position.* For the weight of a body is equivalent to a single force acting through its c. g., and, if this point is supported—that is, if a force equal to the weight, but acting in the opposite direction, is applied at this point—the two forces will balance each other.

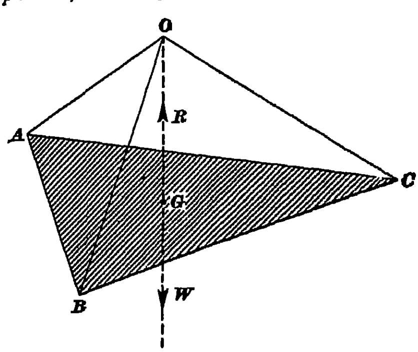


FIG 1

Conversely, *if a body acted on by its own weight and by other forces is in equilibrium, the resultant of the other forces must be vertical, act upwards, and pass through the c. g of the body.* Thus, if the triangular plate ABC, Fig 1, hangs from three strings AO, BO, CO, and is in equilibrium, its weight W, acting through the center of gravity G, must be balanced by the resultant of the reactions of the strings. This resultant must, therefore, be vertical, act upwards, and pass through G.

3. The Center of Gravity of a Body May be Outside the Body.—When several forces act on a body, the line of action of either their resultant or their equilibrant may be altogether outside the body. For example, in Fig. 2, the equilibrant Q of the forces F_1 and F_2 acting on the lever $A_1 A_2$ passes through a point A_q outside the lever. This means

that the only single force that can balance F_1 and F_2 is the force Q acting through A_q; but, as the point A_q is outside the lever, that point must be imagined to be rigidly connected to the lever, in order that Q may act on the latter.

In the same manner, it often happens that the c. g. of a body is outside the body. Thus, the c. g. of the curved

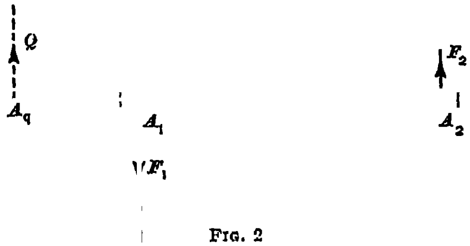


FIG. 2

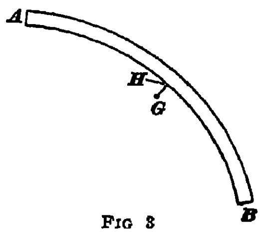


FIG 3

rod AB, Fig. 3, is a point G outside the rod If it is desired to suspend the rod so that it will be in equilibrium in all positions, the point G must be connected to the rod by some means, as by another rod GH. The additional weight of the rod GH alters the position of the c. g. of the whole system; but the change may be made small, and its amount easily calculated, as will be explained presently.

4. **Center of Gravity of a Line.**—Properly speaking, geometrical lines and surfaces have no c. g., since they have no weight. By an extension of the definition, however, every line and surface is said to have a c. g., the expression being taken in the sense now to be explained.

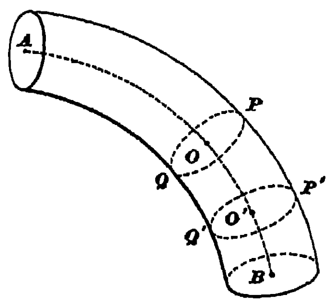


FIG 4

Consider a body AB, Fig. 4, of any form, and a section, as PQ, containing the centers of a number of particles of the body. The resultant of the weights of these particles will pass through a fixed point O, which is the center of the system of parallel forces constituted by those weights. Likewise, O' is the center of the system of parallel

forces constituted by the weights of the particles in the section $P'Q'$. If a sufficient number of sections is taken to include all the particles of the body, and the centers thus found are joined, a line $A\,O\,O'B$ will be obtained, which will be intersected by the resultants of the various groups of particles considered The center of the system formed by these resultants is the c. g of the body, and is said to be also the c g of the line $A\,O\,O'B$.

If the body is homogeneous and of uniform cross-section, all the resultants referred to will be equal, and the number of them acting on any portion of the line $A\,O\,O'B$ will be proportional to the length of that portion. Now, in determining the center of any system of parallel forces, the forces may be replaced by any quantities proportional to them For instance, if the x coordinates of the points of application of two forces F_1 and F_2 are x_1 and x_2, the x coordinate x_c of their center will be given by the formula (see *Analytic Statics*, Part 1)

$$x_c = \frac{F_1 x_1 + F_2 x_2}{F_1 + F_2} = \frac{x_1 + \frac{F_2}{F_1} x_2}{1 + \frac{F_2}{F_1}}$$

If F_2 and F_1 are proportional to two given lengths l_2 and l_1, that is, if $\frac{F_2}{F_1} = \frac{l_2}{l_1}$, we have

$$x_c = \frac{x_1 + \frac{l_2}{l_1} x_2}{1 + \frac{l_2}{l_1}} = \frac{l_1 x_1 + l_2 x_2}{l_1 + l_2}$$

Therefore, $$\frac{F_1 x_1 + F_2 x_2}{F_1 + F_2} = \frac{l_1 x_1 + l_2 x_2}{l_1 + l_2}$$

So, in finding the center of a system of parallel forces whose points of application lie in a line, and whose distribution along any part of the line is proportional to the length of that part, that length may be used instead of the corresponding forces.

5. Center of Gravity of a Plane Area.—Take now a homogeneous prism $ACGE$, Fig. 5. If all the particles

enclosed by a cylindrical surface PQ are considered, the resultant of their weights will be a vertical force whose line of action will meet the upper surface of the prism at some point O. Similarly, other points may be found where the resultants of the weights of vertical rows of particles meet the surface $ABCD$, thus obtaining a system of parallel forces whose points of application may be taken on the plane $ABCD$, and whose center is called the c. g. of the surface $ABCD$. This point is evidently the same as the point where the vertical line through the c. g. of the prism pierces the plane $ABCD$. As the forces acting on any such space as P are proportional to the area of this space, areas may be substituted for forces when dealing with the c. g. of a plane surface.

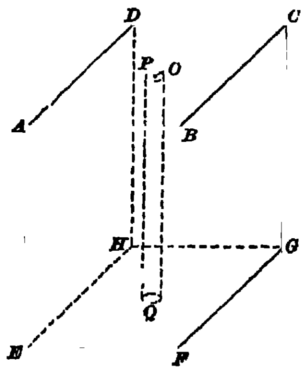


Fig. 5

6. General Definition of the Center of Gravity of Lines and Surfaces.—A line is said to be **homogeneous,** or **of uniform weight,** when parallel forces are distributed along the line in such a manner that the resultant of the forces acting on any part of the line is proportional to the length of that part. The center of the parallel forces thus acting is the **c. g. of the line;** and their resultant is called the **weight of the line.**

The same terms apply to a surface when the parallel forces acting through it (that is, whose lines of action pierce it) are such that the resultant of the forces acting through any part of the surface is proportional to the area of that part. The center of the parallel forces thus acting is the **c. g. of the surface;** and their resultant is called the **weight of the surface.**

7. Static Moment.—Let W be the weight, or any number (length, area) proportional to the weight, of any figure—line, surface, or solid; and let x be the distance of the center

of gravity of the figure from any point or line. Then, the product Wx is called the **static moment** of the figure with respect to that point or line

According to the theory of parallel forces, the static moment of any figure or system of figures about any point or line is equal to the algebraic sum of the moments of the parts of which the figure or system is composed If the distances of the centers of gravity of several areas A_1, A_2, etc from a given point or line are denoted, respectively, by x_1, x_2, etc , the static moments of these areas about the point or line are $x_1 A_1, x_2 A_2$, etc. If the distance of the c g of the system formed by the aggregate of these areas from the given point or line is denoted by x_c, then,

$$x_c(A_1 + A_2 + \ . \ . \ . \ ) = x_1 A_1 + x_2 A_2 + \ . \ . \ .$$

In general, denoting the sum of the areas by ΣA, and the algebraic sum of their static moments by $\Sigma x A$, the following equation may be written:

$$x_c \Sigma A = \Sigma x A;$$

whence,

$$x_c = \frac{\Sigma x A}{\Sigma A}$$

A similar formula applies to lines and volumes. The formula shows that

1. *The algebraic sum of the static moments of the parts of any figure or system of figures about the c. g. of the figure or system of figures, or about a line containing that center, is equal to zero.* For, in this case, $x_c = 0$, and, therefore, $\Sigma x A \ (= x_c \Sigma A) = 0$ Conversely,

2. *If the algebraic sum of the moments of the parts of a figure or system of figures about a point is zero, that point is the c. g. of the figure or system.*

IMPORTANT CASES

SYMMETRICAL FIGURES

8. Definitions.—A figure is **symmetrical** *with respect to a point O* when every straight line passing through that point meets the figure in pairs of points equidistant from the point *O*, the two points of each pair being on opposite sides of the point *O*.

The circle, the ellipse, the sphere, the circular ring, are each symmetrical with respect to its center. This center is called the **geometric center, center of figure,** or **center of symmetry** of the figure in question. A parallelogram is symmetrical with respect to the point of intersection of its diagonals.

9. A figure is **symmetrical** *with respect to a straight line*, called an **axis of symmetry,** if every straight line perpendicular to the first-mentioned line meets the figure in pairs of points equidistant from said line or axis, the two points of each pair being on opposite sides of the axis Thus, a rectangle is symmetrical with respect to either of the lines joining the middle points of two opposite sides. An ellipse is symmetrical with respect to either of its principal axes, and a circle with respect to any diameter. The axis of a right circular cylinder is an axis of symmetry, and so is the axis of a right circular cone.

10. A solid is **symmetrical** *with respect to a plane*, called a **plane of symmetry,** when every straight line perpendicular to the plane meets the solid in pairs of points equidistant from the plane.

11. Center of Gravity of Symmetrical Figures.—It is evident that, *if a figure is symmetrical with respect to a point, a line, or a plane, the c. g. of the figure coincides with the point, or lies in the line or plane, of symmetry, as the case may be.*

Thus, the c. g. of a circle coincides with the center of the circle; the c. g. of a parallelogram coincides with the intersection of the diagonals; the c. g. of an isosceles triangle

lies in the perpendicular from the vertex to the base (at what distance will be seen further on), and the c g of a regular pyramid lies in a plane through the vertex perpendicular to the plane of the base

12. Again, *if a figure has two axes of symmetry, the c. g. will be at the intersection of those axes,* and

If a solid has three planes of symmetry meeting at a point, that point is the c. g. of the solid.

DETERMINATION OF THE CENTER OF GRAVITY BY ADDITION AND SUBTRACTION

13. Addition Method.—If a figure (by which is meant either a line, a surface, or a solid) or system of figures can be divided into parts whose centers of gravity are known, the c. g. of the whole figure or system is easily found by the principles explained in *Analytic Statics*, Part 1.

For example, let it be required to find the c. g of a system consisting of two homogeneous spheres C_1 and C_2, Fig. 6, whose weights are W_1 and W_2, respectively, and whose distance apart, measured between centers, is a. This is equivalent to finding the center of two parallel forces W_1 and W_2 acting through C_1 and C_2. Let this center be G; then,

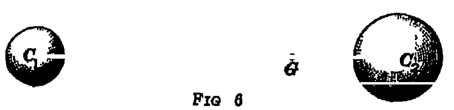


FIG 6

$$W_1 a = (W_1 + W_2) \times G C_2;$$

whence,
$$G C_2 = \frac{W_1 a}{W_1 + W_2}$$

14. Subtraction Method.—If, on the contrary, the c. g. of a system is known, and also the c. g. of all its parts but one, the c. g. of the remaining part is found by the same general principles just referred to, the only difference being that subtraction is used instead of addition.

As an illustration, let it be required to find the c. g. of a figure obtained by cutting from a rectangle $BCDE$, Fig. 7, a rectangular corner $EFHI$ and a circular piece O, the

dimensions being as shown Take ED for the axis of x and EB for the axis of y, and let x' and y' be the coordinates EH' and $H'G'$ of the c. g of the rectangle $BCDE$; x'' and y'', the coordinates EH'' and $H''O$ of the c. g. of the circle, x''' and y''', the coordinates EH''' and $H'''G'''$ of the c g. of the rectangle $EFHI$; and x_c and y_c the coordinates of G, the required c. g. According to Art. **5**, areas may, when dealing with surfaces, be substituted for forces in the equations of moments and of centers of parallel forces. In this

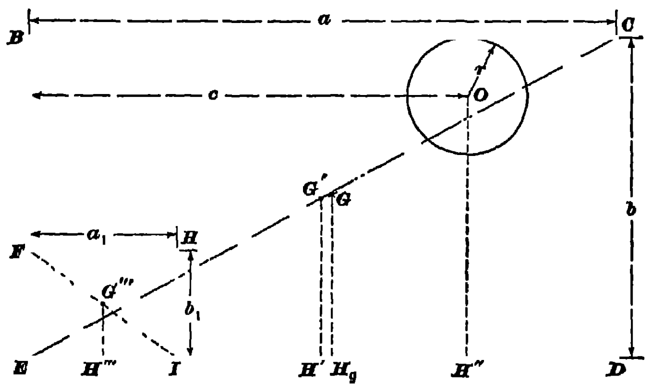


FIG. 7

sense, the moment of $BCDE$ with respect to ED is the same as if the whole area were concentrated at G'; that is, the moment of $BCDE$ with respect to ED is $a\,b\,y'$.

The area A of the figure whose center of gravity is required is $ab - a_1 b_1 - \pi r^2$. The c. g. of $BCDE$ is at the intersection of the two diagonals, or at the middle point of CE, and, therefore, $x' = EH' = \frac{a}{2}$, and $y' = H'G' = \frac{b}{2}$.

Likewise,

$$x'' = EH'' = c,\ y'' = H''O = b - r,$$

and $$x''' = EH''' = \frac{a_1}{2},\ y''' = H'''G''' = \frac{b_1}{2}$$

The formula of Art. **7** now gives

$$x' = \frac{A x_c + a_1 b_1 x''' + \pi r^2 x''}{A + a_1 b_1 + \pi r^2};$$

whence,

$$x_c = \frac{(A + a_1 b_1 + \pi r^2)x' - a_1 b_1 x''' - \pi r^2 x''}{A}$$

$$= \frac{ab \times \frac{a}{2} - a_1 b_1 \times \frac{a_1}{2} - \pi r^2 c}{ab - a_1 b_1 - \pi r^2}$$

$$= \frac{1}{2}\left(\frac{a^2 b - a_1^2 b_1 - 2\pi r^2 c}{ab - a_1 b_1 - \pi r^2}\right)$$

Similarly, $y' = \frac{A y_c + a_1 b_1 y''' + \pi r^2 y''}{ab}$;

whence,

$$y_c = \frac{aby' - a_1 b_1 y''' - \pi r^2 y''}{A} = \frac{1}{2}\left[\frac{ab^2 - a_1 b_1^2 - 2\pi r^2(b - r)}{ab - a_1 b_1 - \pi r^2}\right]$$

15. The subtraction method, just illustrated, is very much simplified when the c. g. of a figure consisting of two parts and the c. g. of one of the parts are known, and it is required to find the c g. of the other part. This special case is illustrated in Fig. 8, where G, the c. g. of the figure $BCDE$, and G_1, the c. g. of the part CED, are known, and it is required to find G_2, the c. g. of the part BCE.

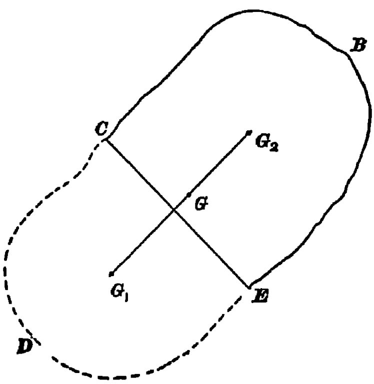


FIG 8

Let total area $= A$; and area $CDE = A_1$. Then (Art. **13**),

$$A_1 \times G_1 G_2 = A \times G G_2;$$

whence, $$G G_2 = G_1 G_2 \times \frac{A_1}{A}$$

EXAMPLE 1—To find the c g. of the channel section represented in Fig 9

SOLUTION.—Owing to the symmetry of the figure with respect to the center line OX, its center of gravity G_c must be on that line. To find the distance $OG_c = x_c$ of G_c from the back of the channel, notice that the section of the latter is the difference between the rectangles $ABCD$ and $A'B'C'D'$, whose centers of gravity G and G' are at the

distances $OG = \frac{5}{2}$ inches, and $OG' = \frac{A'B'}{2} + \frac{5}{8} = \frac{5 - \frac{5}{8}}{2} + \frac{5}{8} = \frac{45}{16}$ inches from AD. Denoting the area of the section by A, and taking moments about AD,

$$A x_c = (AB \times AD) \times OG - (A'B' \times A'D') \times OG'$$
$$= 60 \times \frac{5}{2} - \left(5 - \frac{5}{8}\right)\left(12 - 2 \times \frac{1}{2}\right) \times \frac{45}{16}$$
$$= \frac{150 \times 8 \times 16 - 35 \times 11 \times 45}{8 \times 16}$$

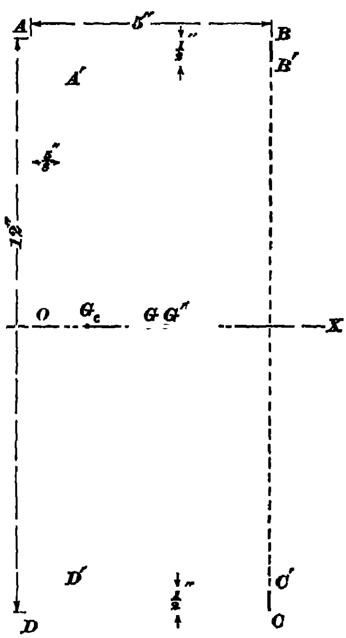


FIG. 9

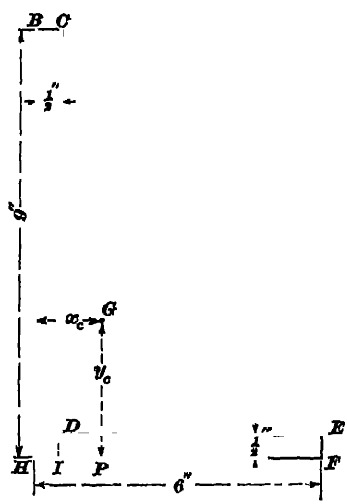


FIG. 10

The area of the section is

$$A = AB \times AD - A'B' \times A'D' = 60 - \frac{35}{8} \times 11 = \frac{8 \times 60 - 35 \times 11}{8}$$

Hence,

$$x_c = OG_c = \frac{150 \times 8 \times 16 - 35 \times 11 \times 45}{8 \times 16} \div \frac{8 \times 60 - 35 \times 11}{8}$$
$$= \frac{1{,}875}{1{,}520} = 1.2335 \text{ in.} = 1\tfrac{7}{32} \text{ in., nearly} \quad \text{Ans.}$$

EXAMPLE 2.—To find the c. g. of the angle section represented in Fig. 10, and having dimensions as shown.

SOLUTION —Produce CD to meet HF at I The section is now divided into two rectangles, $BCIH$ and $DEFI$ The c g of a rectangle being at the point of intersection of the diagonals, its distance from either of two parallel sides is equal to one-half of either of the other two parallel sides Taking moments about HB, and denoting the distances of the c g of the section from HB and HF by x_c and y_c, respectively, and the area of the section by A,

$$A x_c = (HB \times BC) \times \frac{BC}{2} + (DE \times EF) \times \left(\frac{DE}{2} + \frac{1}{2}\right)$$
$$= 9 \times \frac{1}{2} \times \frac{1}{4} + 5\tfrac{1}{2} \times \frac{1}{2} \times \left(\frac{5\frac{1}{2}}{2} + \frac{1}{2}\right) = \frac{9}{8} + \frac{11 \times 13}{16},$$

whence,

$$x_c = HP = \frac{\frac{9}{8} + \frac{11 \times 13}{16}}{A} = \frac{\frac{9}{8} + \frac{11 \times 13}{16}}{9 \times \frac{1}{2} + 5\frac{1}{2} \times \frac{1}{2}} = 1.3879''$$

Moments about HF:

$$A y_c = (HB \times BC) \times \frac{HB}{2} + (DE \times EF) \times \frac{EF}{2}$$
$$= 9 \times \frac{1}{2} \times \frac{9}{2} + 5\tfrac{1}{2} \times \frac{1}{2} \times \frac{1}{4} = \frac{81}{4} + \frac{11}{16},$$

whence,

$$y_c = PG = \frac{\frac{81}{4} + \frac{11}{16}}{9 \times \frac{1}{2} + 5\frac{1}{2} \times \frac{1}{2}} = \frac{81 + \frac{11}{4}}{29} = 2\ 8879 \text{ in} \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE

1 Find the center of gravity G of the area obtained by taking the rectangles BF and DH, Fig 11, from the rectangle $ABCD$ The

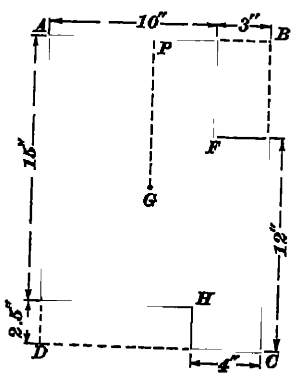


FIG 11

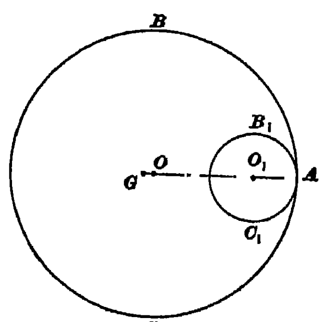


FIG. 12

dimensions are as shown.

Ans. $\begin{cases} AP = 6\ 3011 \text{ in.} = 6\frac{5}{16} \text{ in., nearly} \\ PG = 8.3800 \text{ in.} = 8\frac{3}{8} \text{ in., nearly} \end{cases}$

2 Find the c g of the area obtained by cutting off the circle $A B_1 C_1$, Fig 12, whose radius $O_1 A$ is 3 inches, from the circle $A B C$, whose radius $O A$ is 10 inches Ans $O G = .69231$ in $= \frac{11}{16}$ in., nearly

3 Find the c g of the angle section represented in Fig 13

Ans $\begin{cases} O P = .92500 \text{ in.} = \frac{15}{16} \text{ in., nearly} \\ P G = 2.1750 \text{ in.} = 2\frac{3}{16} \text{ in., nearly} \end{cases}$

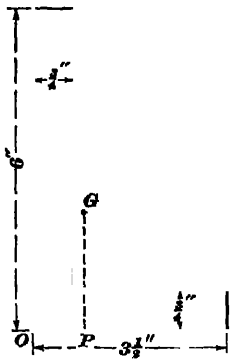


FIG 13

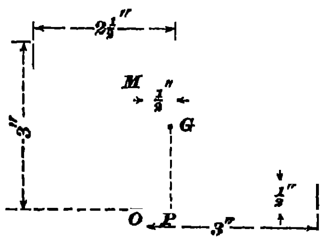


FIG 14

4 Find the c g of the Z section represented in Fig 14

NOTE —In taking moments about $O M$, remember that the moments of areas on opposite sides have opposite signs

Ans $\begin{cases} O P = .41667 \text{ in.} = \frac{13}{32} \text{ in., nearly} \\ P G = 1.4167 \text{ in.} = 1\frac{13}{32} \text{ in., nearly} \end{cases}$

CENTER OF GRAVITY OF POLYGONS

NOTE —In some of the articles that follow, formulas and rules are given without explaining how they are obtained This is done whenever the processes involved require the use of advanced mathematics, or, being elementary, are too long and complicated to be given in connection with this instruction

16. A Fundamental Principle.—*If a straight line divides a plane figure in such a manner that every line parallel to a fixed direction meets the perimeter of the figure at one point on each side of the first-mentioned line and is bisected by said line, the c. g. of the figure lies on that line.*

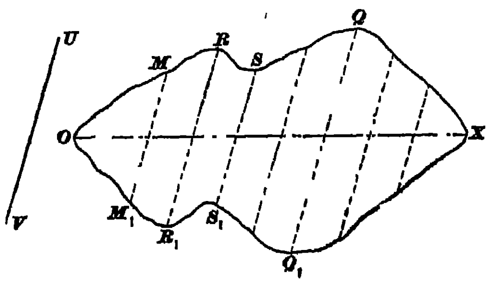


FIG. 15

Let $O X$, Fig. 15, be a line dividing the figure $O Q X Q_1$ into two parts, in such a manner that all lines,

as QQ_1, RR_1, etc., parallel to the fixed line UV are bisected at their point of intersection with OX. Then, according to the principle just stated, the c. g. of the area $OQXQ_1$ lies in the line OX.

17. Triangle.—Let ABC, Fig. 16, be any triangle. Draw AA' from A to the middle point A' of the opposite side. Any line, as RS, parallel to BC is bisected at its intersection J with AA'. Therefore, according to the proposition of the preceding article, the c. g. of the triangle lies on AA'. For a similar reason, the c. g. must lie on BB', joining the vertex B and the middle point B' of the opposite side AC. Therefore, the c. g. of the triangle is at the intersection of the lines AA' and BB', or of either of them with the line CC' from C to the middle of AB. It is shown in geometry that the distance of the point G, where AA', BB', and CC' meet, from any of the vertexes is equal to two-thirds the length of the line joining that vertex to the middle point of the opposite side, or $AG = \frac{2}{3}AA'$, $BG = \frac{2}{3}BB'$, $CG = \frac{2}{3}CC'$. The lines AA', BB', CC' are called the **median lines**. Therefore,

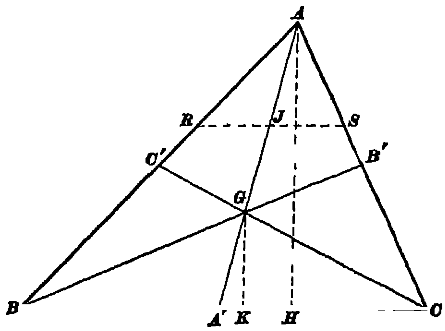


FIG. 16

The c. g. of a triangle lies at the intersection of the median lines, and its distance from any vertex is equal to two-thirds the length of the median line from that vertex to the opposite side.

18. *The perpendicular distance from the c. g. of a triangle to any of the sides is equal to one-third the altitude of the triangle, when that side is taken as the base.*

For, drawing AH and GK perpendicular to BC, Fig. 16, two similar triangles $AA'H$ and $GA'K$ are formed, which give

$$\frac{AA'}{AH} = \frac{GA'}{GK};$$

whence, bearing in mind that $GA' = \frac{1}{3}AA'$ or $\frac{GA'}{AA'} = \frac{1}{3}$,

$$GK = \frac{GA'}{AA'} \times AH = \frac{1}{3}AH$$

19. Sometimes, the distances of the vertexes of a triangle from an axis or line of reference are given, and it is required to find the distance of the c. g. from the same line. In Fig. 17, let the distances of the vertexes of the triangle $A_1A_2A_3$ from the line $X'X$ be y_1, y_2, y_3. The c. g. of the triangle is on the median A_1M, and, as already explained, $GM = \frac{1}{3}A_1M$. Let the distance GP of the c. g. from $X'X$ be denoted by y_c. Draw MN perpendicular and MB parallel to $X'X$. The figure gives

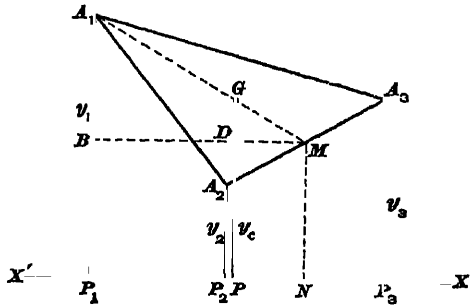


FIG 17

$$y_c = GP = DP + GD = MN + GD \qquad (1)$$

In the trapezoid $A_2P_2P_3A_3$, the line MN joins the middle point of A_2A_3 and P_2P_3, therefore,

$$MN = \frac{1}{2}(y_2 + y_3) \qquad (2)$$

In the similar triangles A_1MB and GMD,

$$\frac{GM}{A_1M} = \frac{GD}{A_1B}, \text{ that is, } \frac{\frac{1}{3}A_1M}{A_1M} = \frac{GD}{A_1B};$$

whence,

$$GD = \tfrac{1}{3}A_1B = \tfrac{1}{3}(A_1P_1 - MN) = \tfrac{1}{3}[y_1 - \tfrac{1}{2}(y_2 + y_3)] \qquad (3)$$

Substituting in equation (1) the values of MN and GD given by equations (2) and (3), and reducing,

$$y_c = \tfrac{1}{3}(y_1 + y_2 + y_3)$$

This formula is perfectly general, provided that due attention is paid to the signs of the coordinates. That is, distances

measured on one side of the reference line should be considered positive, and those measured on the opposite side, negative. Thus, if the vertex A_2 lies below $X'X$, and y_1 and y_2 are considered positive, y_3 should be considered negative.

20. Trapezoid.—Let $BCDE$, Fig 18, be any trapezoid, having the bases $BE = b_1$, $CD = b_2$; altitude $DH = h$, median line $M_1M_2 = m$ The median line M_1M_2 (that is, a line through the middle points of the bases) bisects every line parallel to the bases; therefore, G, the c. g of the trapezoid, must lie on that line (Art **16**). It is necessary, therefore, only to find the distance, as GP_1, of G from either base.

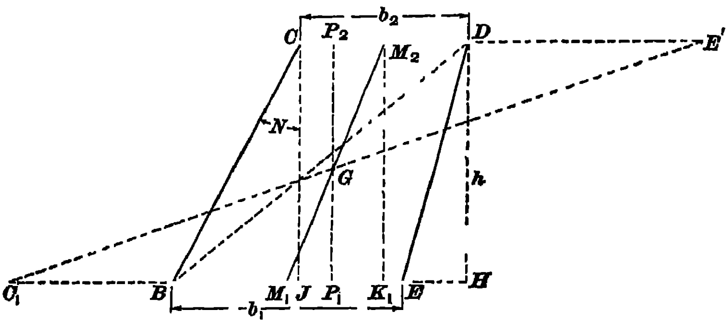


FIG 18

Draw BD; this line divides the trapezoid into two triangles having the same altitude h, and b_1 and b_2 for their bases. The distance of the c. g. of BCD from CD is $\frac{1}{3}h$ (Art. **18**), and its distance from BE is $h - \frac{1}{3}h = \frac{2}{3}h$. The distance of the c. g. of BDE from BE is $\frac{1}{3}h$. Denoting the area of the trapezoid by A, and taking moments about BE,

$$A \times GP_1 = \text{area } BCD \times \tfrac{2}{3}h + \text{area } BDE \times \tfrac{1}{3}h$$

$$= \tfrac{1}{2}b_2 h \times \tfrac{2}{3}h + \tfrac{1}{2}b_1 h \times \tfrac{1}{3}h = (b_1 + 2b_2)\frac{h^2}{6};$$

whence,

$$GP_1 = \frac{(b_1 + 2b_2)\frac{h^2}{6}}{A} = \frac{(b_1 + 2b_2)\frac{h^2}{6}}{\frac{1}{2}(b_1 + b_2)h} = \frac{h}{3}\left(\frac{b_1 + 2b_2}{b_1 + b_2}\right) \qquad (1)$$

A similar value may be found for GP_2, by simply interchanging b_1 and b_2 in this formula.

Draw M_2K_1 perpendicular to the bases. The similar triangles $M_2M_1K_1$ and GM_1P_1 give

$$\frac{M_2M_1}{GM_1} = \frac{M_2K_1}{GP_1}, \text{ or } \frac{m}{GM_1} = \frac{h}{\frac{h}{3}\left(\frac{b_1+2b_2}{b_1+b_2}\right)} = \frac{3(b_1+b_2)}{b_1+2b_2};$$

whence,
$$GM_1 = \frac{m}{3}\left(\frac{b_1+2b_2}{b_1+b_2}\right) \qquad (2)$$

which gives the distance from the middle point of b_1 to the c. g., measured along the median line. A similar expression may be found for GM_2, by simply interchanging b_1 and b_2.

The distance BP_1 is found by the following formula, in which N is the angle between CB and CJ, the latter line being perpendicular to the bases:

$$BP_1 = \tfrac{1}{3}\left[b_1 + b_2 + \frac{(b_1+2b_2)\,h\tan N - b_1b_2}{b_1+b_2}\right] \qquad (3)$$

21. There is another formula that is often useful for the determination of the c. g. of a trapezoid. Let the non-parallel sides EB and DC, Fig. 19, meet at V. As before, M_1M_2 is the line through the middle points of the bases.

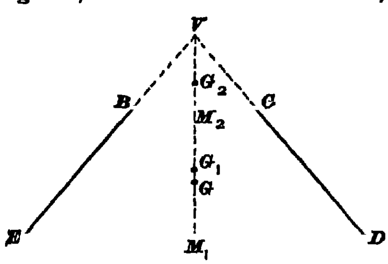


Fig 19

From geometric principles, it is known that this line passes through V. Let G_1 be the c.g. of the triangle DVE, G_2 the c. g. of CVB, and G that of the trapezoid $BCDE$. Also, let $VM_1 = a_1$, $VM_2 = a_2$, $VG = a$. As explained in Art. **17**, $VG_1 = \frac{2}{3}a_1$, and $VG_2 = \frac{2}{3}a_2$. Taking moments about G, we have,

$$\text{area } DVE \times G_1G = \text{area } CVB \times G_2G;$$

whence,

$$\frac{G_1G}{G_2G} = \frac{\text{area } CVB}{\text{area } DVE} = \frac{VM_2^2}{VM_1^2}$$

since the areas of two similar triangles are proportional to the squares of their homologous sides. In terms of the

quantities a_1, a_2, and a, the preceding proportion may be written:

$$\frac{a - \frac{2}{3}a_1}{a - \frac{2}{3}a_2} = \frac{a_2^2}{a_1^2}$$

Solving this equation for a, the following result is finally obtained:

$$a = \frac{2}{3}\,\frac{a_1^3 - a_2^3}{a_1^2 - a_2^2}$$

22. Trapezoid: Graphic Solution.—Produce CD to E', Fig. 18, making $DE' = b_1$, draw $E'G$, and produce it to its intersection C_1 with EB produced. The similar triangles GM_2E' and GM_1C_1 give:

$$\frac{M_1C_1}{GM_1} = \frac{M_2E'}{GM_2};$$

whence, $$M_1C_1 = \frac{GM_1}{GM_2} \times M_2E' \qquad (1)$$

The value of GM_1 is given by formula **2** of Art. **20**, and the value of GM_2 is given by the same formula, by interchanging b_1 and b_2. Therefore,

$$\frac{GM_1}{GM_2} = \frac{\dfrac{m}{3} \times \dfrac{b_1 + 2b_2}{b_1 + b_2}}{\dfrac{m}{3} \times \dfrac{b_2 + 2b_1}{b_2 + b_1}} = \frac{b_1 + 2b_2}{b_2 + 2b_1} = \frac{\frac{1}{2}b_1 + b_2}{\frac{1}{2}b_2 + b_1}$$

By construction, $M_2E' = M_2D + DE' = \frac{1}{2}b_2 + b_1$. Substituting these values in (1),

$$M_1C_1 = \frac{\frac{1}{2}b_1 + b_2}{\frac{1}{2}b_2 + b_1} \times (\tfrac{1}{2}b_2 + b_1) = \tfrac{1}{2}b_1 + b_2 = M_1B + b_2;$$

therefore, $$BC_1 = b_2$$

Hence, the following construction for finding the c. g. of any trapezoid.

Join the middle points of the bases. Produce the bases in opposite directions, making the prolonged part of each base equal to the other base. Draw a line joining the extremities of the prolonged segments. The point where this line intersects the median line is the required c. g.

23. Any Quadrilateral.—Let $BCDE$, Fig. 20, be any quadrilateral. Draw the diagonals BD and EC, and find the

middle point M of one of them, in this case EC. Take on DB the distance $BN = DI$, and draw MN. The c. g. of the quadrilateral is a point G on MN, obtained by taking on MN a distance $MG = \frac{1}{3} MN$.

EXAMPLE 1.—To find the c. g. of the channel section represented in Fig. 21, the dimensions being as shown.

NOTE.—The section being symmetrical, the dimensions of the lower part are the same as the corresponding dimensions of the upper part, and need not be given.

SOLUTION.—The c. g. lies on the line of symmetry OQ drawn through the middle point of, and perpendicular to, BJ. The only thing to be determined is the distance $OG = x_c$.

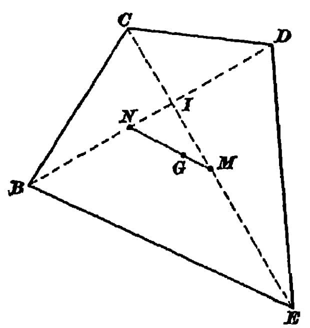


FIG. 20

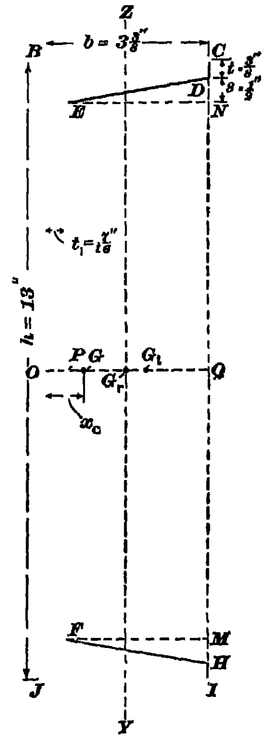


FIG. 21

In complicated cases like this, it is often convenient to find a general formula first, and then substitute the numerical values. In order to do this, the dimensions will be represented by the letters written beside the figures on the diagram. Since the channel section is the difference between the rectangle $BCIJ$ and the trapezoid $DEFH$, then, denoting the area of the rectangle by R, the area of the trapezoid by T, that of the section by A, and their centers of gravity by G_r, G_t, and G, respectively,

$$x_c = OG_r - GG_r = \tfrac{1}{2}b - GG_r \qquad (1)$$

Taking moments about a line YZ drawn parallel to BJ through the center G_r of the rectangle $BCIJ$, we have, since the moment of R about G_r is 0,

$$GG_r = \frac{T \times G_r G_t}{A} \qquad (2)$$

Now,

$$T = \tfrac{1}{2}(EF + DH) \times PQ = \tfrac{1}{2}[(h - 2s - 2t) + (h - 2t)](b - t_1)$$
$$= (h - s - 2t)(b - t_1)$$

and (formula 1 of Art 20),

$$G_r G_t = QG_r - QG_t = \tfrac{1}{2}b - \frac{b - t_1}{3} \times \frac{(h - 2t) + 2(h - 2s - 2t)}{(h - 2t) + (h - 2s - 2t)}$$
$$= \tfrac{1}{2}b - \frac{b - t_1}{3} \times \frac{3h - 4s - 6t}{2(h - s - 2t)}$$

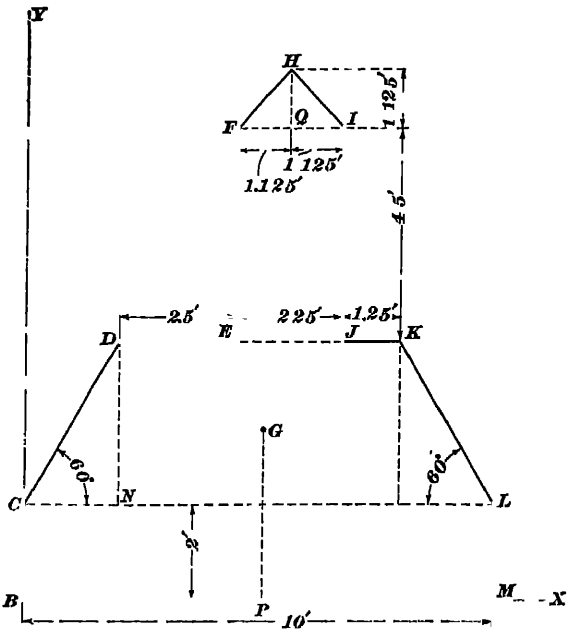


FIG 22

Substituting the values of T and $G_r G_t$ in (2),

$$GG_r = \frac{(h - s - 2t)(b - t_1)\left[\frac{1}{2}b - \frac{b - t_1}{3} \times \frac{3h - 4s - 6t}{2(h - s - 2t)}\right]}{A}$$
$$= \frac{b - t_1}{6A}[3b(h - s - 2t) - (b - t_1)(3h - 4s - 6t)]$$
$$= \frac{b - t_1}{6A}[3b(h - s - 2t) - 3(b - t_1)(h - s - 2t) + s(b - t_1)]$$
$$= \frac{b - t_1}{6A}[3t_1(h - s - 2t) + s(b - t_1)]$$
$$= \frac{(b - t_1)[t_1(h - s - 2t) + \frac{1}{3}s(b - t_1)]}{2A}$$

Substituting the values of OG_r and GG_r in (1),

$$x_c = \tfrac{1}{2}b - \frac{(b-t_1)[t_1(h-s-2t)+\frac{1}{3}s(b-t_1)]}{2A}$$

The dimensions, expressed in sixteenths of an inch, are $b = 54$; $h = 208$, $t = 6$, $s = 8$, $t_1 = 7$. Therefore,

$$A = bh - (b-t_1)(h-s-2t) = 54 \times 208 - (54-7)(208-8-12);$$

and

$$x_c = \frac{54}{2} - \frac{(54-7)[7(208-8-12)+\frac{8}{3}(54-7)]}{2[54\times 208-(54-7)(208-8-12)]}$$

$$= 27 - \frac{47\left(7\times 188+\dfrac{8\times 47}{3}\right)}{2\times 2{,}396} = 27 - 14.137 = 12.863 \text{ sixteenths}$$

$= .804$ in., or, approximately, $x_c = \frac{13}{16}$ in. Ans.

Example 2.—To find the c. g. of the plane figure represented in Fig. 22, the dimensions being as shown.

Solution.—The figure may be divided into the two rectangles $BCLM$ and $EFIJ$, the trapezoid $CDKL$, and the isosceles triangle FHI. Moments will be taken about BX and BY. The altitude of the trapezoid is $DN = CN \tan 60° = \frac{1}{2}(CL - DK)\tan 60°$ $= 3.4641$ ft. The operations are given in the following table, which needs no explanation.

Figure	Area Square Feet	Lever Arm With Respect to BX	Lever Arm With Respect to BY	Moment About BX	Moment About BY
$BCLM$	20.0000	1.0000	5.0000	20.0000	100.0000
$CDKL$	27.7128	$\left\{2+\frac{DN}{3}\times\frac{CL+2DK}{CL+DK} = 3.5877\right.$	5.0000	99.4252	138.5640
$EFIJ$	10.1250	$\left\{\frac{4.5}{2}+DN+CB = 7.7141\right.$	$\left\{\frac{2.25}{2}+ED+CN = 5.6250\right.$	78.1053	56.9531
FHI	1.2656	$\left\{\frac{HQ}{3}+FE+DN+CB = 10.3391\right.$	5.6250	13.0852	7.1190
	59.1034			210.6157	302.6361

Having the area and the resultant moments,

$$x_c = BP = \frac{302.6361}{59.1034} = 5.1205 \text{ ft. Ans.}$$

$$y_c = PG = \frac{210.6157}{59.1034} = 3.5635 \text{ ft. Ans.}$$

EXAMPLE FOR PRACTICE

Find the area and the c. g. of the irregular T section represented in Fig. 23

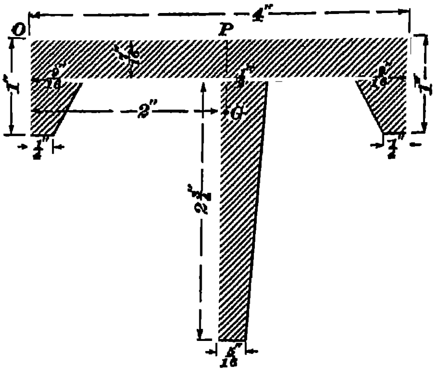


Fig. 23

$$\text{Ans.}\begin{cases} A = 3.3242 \text{ sq. in.} \\ OP = 2.0695 \text{ in.} \\ PG = .7826 \text{ in.} \end{cases}$$

CENTER OF GRAVITY OF AREAS BOUNDED BY CIRCULAR ARCS

24. Circular Sector.—Let $OPSV$, Fig. 24, be a circular sector, having the radius $OP = r$, and the central angle $POV = L$. As the sector is symmetrical with respect to the line OS, bisecting the angle L, the c. g. lies on that line, and it is only necessary to determine the distance OG of the c. g. from the center of the sector. This distance is given by the formula

$$y_c = OG = \frac{240\, r \sin \frac{1}{2} L}{\pi L}$$

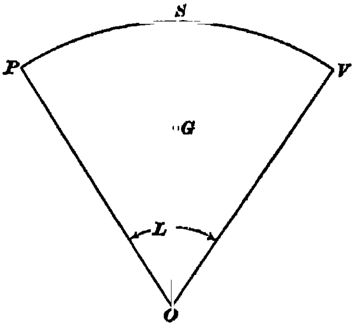


Fig. 24

In the denominator of this formula, the angle L is expressed in degrees and decimals of a degree.

25. For the sectors whose arcs are, repectively, a semicircumference, a quadrant, and a sextant, the preceding formula takes the following special forms:

$$\text{Semicircle, } y_c = \frac{4\, r}{3\, \pi} = .4244\, r \qquad (1)$$

$$\text{Quadrant, } y_c = \frac{8\, r \sin 45°}{3\, \pi} = .6002\, r \qquad (2)$$

$$\text{Sextant, } y_c = \frac{4\, r \sin 30°}{\pi} = .6366\, r \qquad (3)$$

26. Circular Segment.—Let BCD, Fig. 25, be a circular segment, having the radius $OB = r$, and the central angle $BOD = L$, expressed in degrees and decimals of a degree. Here, G, the c. g. of the segment, lies on the line of symmetry OC. The distance $y_c = OG$ is found by the following formula:

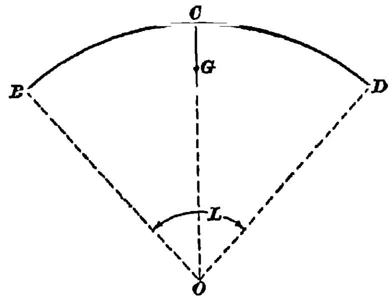


FIG 25

$$y_c = OG = \frac{240\, r \sin^2 \frac{1}{2} L}{\pi L - 180 \sin L} \qquad (1)$$

If the length of the chord BD is denoted by c and the area of the segment BCD by A, the preceding formula can be reduced to

$$y_c = OG = \frac{c^3}{12 A} \qquad (2)$$

27. Circular Trapezoid, or Flat Ring.—In Fig. 26, let $BCDEHI$ be a circular trapezoid, or flat ring, having the radii r_1 and r_2, and the central angle L, expressed in degrees and decimals of a degree. The c. g. lies on the line of symmetry OC, and its distance y_c from the center O of the ring is given by the formula:

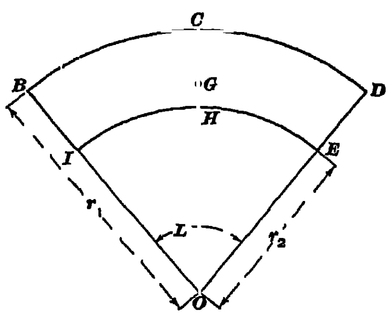


FIG. 26

$$y_c = \frac{240 \sin \frac{1}{2} L}{\pi L} \times \frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}$$

EXAMPLE —To find the c. g. of the area represented in Fig 27

SOLUTION —Take the center line OY, bisecting the two circular arcs, for the axis of y, and the perpendicular OX, through the center of the arcs, for the axis of x As usual, x_c and y_c are the coordinates of G, the required c g , and A is the area of the figure The diagram gives A = rectangle $BFHT$ + segment CDE − segment KJI. The

moment of A with respect to OX is equal to the algebraic sum of the moments of these three areas To find the area of the rectangle, KI must first be found, since TK and IH are given To find the

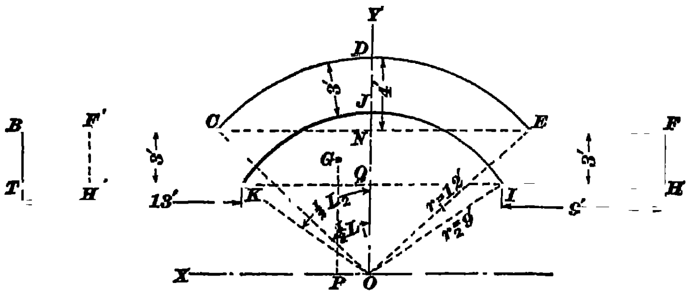


FIG 27

areas of the segments and their lever arms, the angles L_1 and L_2 must be determined The figure gives, expressing angles to the nearest minute,

$$KI = 2\,KQ = 2\sqrt{r_2^2 - OQ^2} = 2\sqrt{r_2^2 - (r_2 - JQ)^2}$$

$$= 2\sqrt{r_2^2 - (r_2 - 4)^2} = 14\ 967 \text{ ft}$$

$$\sin \tfrac{1}{2} L_2 = \frac{\frac{1}{2} KI}{r_2}, \text{ whence } L_2 = 112° \ 30'$$

$$\sin \tfrac{1}{2} L_1 = \frac{CN}{r_1} = \frac{\sqrt{r_1^2 - (r_1 - 4)^2}}{r_1}, \text{ whence } L_1 = 96° \ 23'$$

Having all the required elements, the coordinates x_c and y_c are found by the usual methods, with which the student is now supposed to be perfectly familiar. In finding x_c, however, the operations will be much shortened by observing that, if a distance $KH' = IH$ is taken, and the line $H'F'$ drawn perpendicular to TK, the figure at the right of $F'H'$ is symmetrical with respect to OY, and its moment about OY is consequently zero Therefore, the moment of the whole area about OY is simply the moment of the rectangle $BF'H'T$, and, hence,

$$x_c = \frac{BT \times TH' \times \left(\frac{TH'}{2} + H'Q\right)}{A}$$

The results, which should be verified by the student, are·

$A = 118\ 35$ sq. ft. Ans.

$x_c = OP = 1\ 8741$ ft. Ans

$y_c = PG = 7\ 7607$ ft. Ans.

EXAMPLES FOR PRACTICE

1. Find the c g of a circular sector whose radius is 10 feet and whose central angle is 45°

Ans $y_c = 6\ 497$ ft

2 Find the c g of a segment whose chord is 8 inches and whose radius is 5 inches Ans. $y_c = 3\ 815$ in.

3 Find the c g of a circular trapezoid whose radii are 6 inches and 3 inches, and whose shorter chord (as $E\,I$, Fig. 26) is 3 inches. Ans. $y_c = 4\ 456$ in.

4. Find the c g of the half circular segment $B\,C\,D$ (Fig 28).

Ans. $\begin{cases} x_c = 2.839 \text{ in} \\ y_c = 13.798 \text{ in.} \end{cases}$

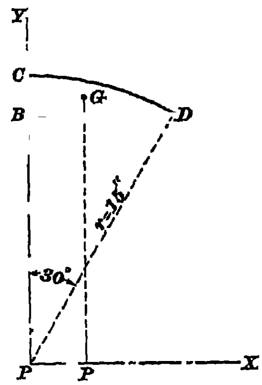


Fig 28

CENTER OF GRAVITY OF A PLANE AREA BOUNDED BY AN IRREGULAR CURVE

28. Approximate Analytic Method.—To determine the c. g. of a figure, as $B\,C\,D\,E$, Fig. 29, having an irregular contour, proceed as follows:

Draw two lines of reference $O\,X$ and $O\,Y$ perpendicular to each other in any convenient positions; preferably, one of the lines, as $O\,X$, should as nearly as possible bisect the area. Divide $O\,X$ into a sufficient number of parts, so that, by erecting perpendiculars at the points of division, the part of the perimeter of the curve intercepted by two consecutive ordinates, as H_1H_2, and I_1I_2, may be treated as a straight line, and the corresponding strip $H_1H_2I_2I_1$, as a trapezoid. Measure the distances $H\,I$, $I\,J$, etc., and the

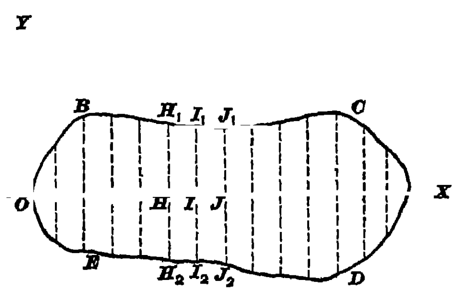


Fig. 29

ordinates $H_1 H_2$, $I_1 I_2$, etc. Find the distances of the centers of gravity of the strips, considered as trapezoids, from the lines OX and OY. Treat the area of the whole figure $BCDE$ as equal to the sum of the areas of the trapezoids, and apply the method of moments, as usual That is, if the sum of the areas of the trapezoids is denoted by ΣT, and the sum of their moments about OX and OY by ΣTy and ΣTx, respectively, the coordinates x_c and y_c of the c. g. of the whole figure are given by the formulas

$$x_c = \frac{\Sigma Tx}{\Sigma T} \qquad (1)$$

$$y_c = \frac{\Sigma Ty}{\Sigma T} \qquad (2)$$

Although this method is only an approximation, it is sufficiently close for almost all practical purposes. The c. g. of each trapezoid may usually be taken at the middle point of its median line

29. Experimental Methods.—A very convenient method of finding the c. g. of an irregular figure (or of any other figure) consists in drawing the figure to scale on a piece of cardboard of uniform thickness, then cutting off the remaining part of the cardboard and balancing the part thus left on a knife edge in two positions.

In Fig. 30, let $BB'CC'$ be the piece cut out of the cardboard, having the form of the figure whose c. g. is required; let BC and $B'C'$ be two positions of the knife edge, for each of which the piece of cardboard remains in equilibrium. Then, their intersection G is evidently the required c. g.

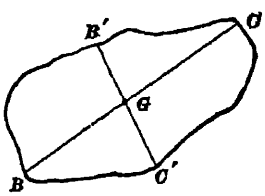


FIG 30

This method may be used not only for irregular figures, but also for those figures the determination of whose center of gravity leads to complicated formulas. The method, however, is not very accurate, and, as a check, the piece of cardboard should be balanced in more than two positions—say in six or eight. This will give an idea of the degree of accuracy attained.

30. The method just described is also very convenient for determining the distance of the c g. of a symmetrical body from a plane perpendicular to a plane or axis of symmetry. A locomotive, for instance, is symmetrical with respect to a plane midway between and parallel to, the axis of the cylinders. The distance of the c. g from either end of the locomotive (that is, from a plane through either end perpendicular to the axes of the cylinders) is found by balancing the locomotive on a horizontal rod perpendicular to the plane of symmetry, passed through rings attached to the locomotive and symmetrically located with respect to that plane. The position of the rod and rings is changed until the locomotive is found to remain balanced (that is, with its two ends at the same level) when suspended. The distance of the rod, when the locomotive is thus balanced, from either end of the locomotive, is, approximately, the distance required.

31. Still another method for finding the c. g. of a plane figure forming the faces of a thin plate of uniform thickness consists in suspending the plate by a string and marking on either face a vertical line that is the prolongation of the direction of the string; then suspending the plate in a different position (that is, tying the string to a different point in the plate) and marking a line similar to the one marked before. The intersection of the two lines is the required c. g.

This method can be conveniently used for determining the c. g. of any body when that center is outside the body (as an angle section, a bent rod, etc.). Here, when the body is suspended in one position by a string, a line containing the c. g. is obtained by making two marks on the body: one at the point where the string is fastened and another directly under it. By tying the string to another point of the body, another line is determined, whose intersection with the first gives the required c. g.

CENTER OF GRAVITY OF SOLIDS

32. Right Cylinder or Prism.—The c g. of a homogeneous right cylinder or prism evidently coincides with the middle point of the line joining the centers of the bases.

33. Right Cone or Regular Pyramid.—The c. g. of any right cone or regular pyramid lies on the perpendicular from the vertex to the base (line joining vertex with center of base) at a distance from the vertex equal to three-quarters of the length of that perpendicular.

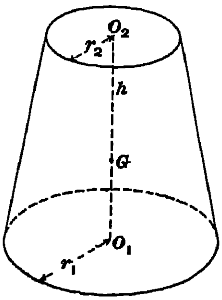


FIG 31

34. Conical Frustum.—The c. g. of a conical frustum, Fig. 31, lies on the line joining the centers of its bases. Its distance from the lower base is given by the formula

$$y_c = O_1 G = \frac{h}{4} \times \frac{(r_1 + r_2)^2 + 2 r_2^2}{(r_1 + r_2)^2 - r_1 r_2}$$

in which r_1 = radius of lower base;
r_2 = radius of upper base;
h = altitude of frustum.

COPLANAR NON-CONCURRENT FORCES

COUPLES

DEFINITIONS—EFFECT OF A COUPLE

35. A **statical couple**, or simply a **couple**, has already been defined as a system of two equal non-collinear parallel forces having opposite directions.

In Fig. 32, the forces F_1 and F_2 constitute a couple acting on the body BCD. Although F_1 and F_2 are equal in magnitude, they are denoted by different letters for convenience in referring to their lines of action.

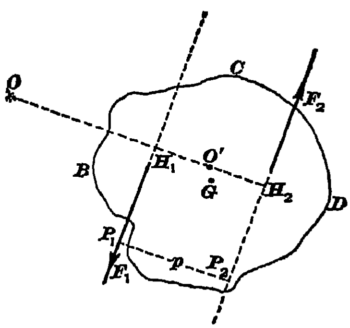


FIG. 32

36. The **lever arm**, or simply the **arm**, of a couple is the perpendicular distance ($P_1 P_2 = p$) between the lines of action of the two forces constituting the couple.

37. The **plane** of a couple is the plane determined by the lines of action of the two forces constituting the couple.

38. The **axis** of a couple is any line perpendicular to the plane of the couple. Such will be the meaning given to the term here, although some writers use it in a different sense. It follows from this definition that all couples whose planes are either coincident or parallel have the same axis.

39. **Coaxial couples** are couples having the same axis. Couples not having the same axis are called **non-coaxial.**

40. **Resultant Moment of the Forces of a Couple.** *The resultant moment of the two forces of a couple about any*

point in their plane is constant and equal to the moment of either force about any point on the line of action of the other force.

This is easily shown Paying due attention to signs, the resultant moment of the two forces F_1 and F_2 about any point O, Fig 32, is

$$- F_2 \times H_2 O + F_1 \times H_1 O = - F_2 (H_2 O - H_1 O) = - F_2 \times H_1 H_2 = - F_2 p = - F_1 p$$

For a point O' between the lines of action of the forces, the resultant moment is

$$- F_2 \times H_2 O' - F_1 \times H_1 O' = - F_2 (H_2 O' + H_1 O') = - F_2 p = - F_1 p$$

41. The constant resultant moment of the forces of a couple about any point in their plane is called the **moment of the couple**, and is numerically equal to the product of either force by the arm of the couple.

42. Notation.—A couple is expressed either by its moment (for all coaxial couples having the same moment are equivalent, as will be shown presently), or by writing its two forces in parenthesis with the arm between; thus, (F_1, p, F_2). The latter is a more convenient form of expression for some purposes, and will be often used here.

43. Effect of a Couple.—*The effect of a couple acting on a rigid body not acted on by other forces is to turn the body about an axis passing through its c g.*

A full demonstration of this proposition cannot be given here, as it would be necessary to make use of some kinetic principles that have not yet been explained. Moreover, it is to be observed that, for the purposes of statics, it is not necessary to know what the effect of a couple is, for all the theorems relating to the equilibrium and equivalence of couples can be stated and proved without any reference to what the effect would be if the couples were unbalanced. The foregoing principle, however, has been stated here, in

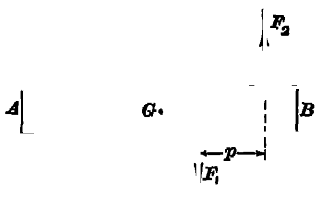


FIG 33

order to caution the student against a common error prevalent among beginners. If the couple (F_1, p, F_2), Fig 33, acts on the body AB, its tendency is to turn the body about an axis passing through its center of gravity G, not about an axis or point situated between the two forces.

44. Direction and Sign of a Couple.—The sign of a couple will here be treated as positive or negative according as the moment of either force about a point in the line of action of the other is positive or negative (see *Analytic Statics*, Part 1). In both Fig. 32 and Fig. 33, the moment of F_1 about any point in the line of action of F_2, or of F_2 about any point in the line of action of F_1, is negative, and the moment of the couple is considered negative.

The motion that the couple (F_1, p, F_2), Fig 32, tends to produce is evidently the reverse in direction of the motion of the hands of a watch with its face placed upwards and its center at H_1 or H_2. This motion is said to be *counter-clockwise*, or *left-handed* Motion similar to that of the hands of a watch is said to be *clockwise*, or *right-handed*. Clockwise motion is said to have a right-handed direction; counter-clockwise motion, a left-handed direction.

It will be noticed that the sign of a couple is positive when the couple tends to produce clockwise motion; otherwise, however, the couple is negative. This distinction is of value only when couples are to be combined by algebraic addition.

The direction and sign of a couple can be very readily determined by imagining the arm of the couple to be a line rotating about its center so that each extremity follows the direction of the force applied at it. Thus, in Fig. 32, if we conceive $H_1 H_2$ to begin to rotate about its center so that H_1 will follow the direction of F_1, and H_2 the direction of F_2, it will be seen that the rotation will be counter-clockwise. The couple is, therefore, a left-handed couple, and its sign is negative.

EQUIVALENCE AND EQUILIBRIUM OF COAXIAL COUPLES

45. Equilibrant and Equivalent Couples.—Since a couple cannot be replaced by a single force, it follows that no single force can balance a couple In order to balance a couple, another couple must be opposed to it. Either couple is called the **equilibrant** of the other.

Two couples are **equivalent** when they can each be balanced by one and the same couple—that is, when they have the same equilibrant.

46. Equilibrium of Two Coaxial Couples.—*Two coaxial couples balance each other if they have equal but opposite moments—that is, if their moments are numerically equal but have opposite directions (or signs)*

DEMONSTRATION —The demonstration of this principle is given below Although it is not essential that it be learned or even read, it affords a very useful and interesting exercise in the composition and resolution of forces It will be necessary to distinguish three cases, as follows

Case I.—*When the two couples are in the same plane and the forces of one are parallel to the forces of the other*

Let the couples be (F_1, p, F_2) and (F_1', p', F_2'), Fig. 34 It is assumed that, with the usual notation as to signs, $F_1 p = -F_1' p'$, or $F_1 p + F_1' p' = 0$ On the other hand, the algebraic sum of the four forces F_1, F_2, F_1', F_2' is equal to zero, or $\Sigma F = 0$ Therefore, the four forces, or the two couples, satisfy the two necessary conditions of equilibrium of parallel forces (*Analytic Statics*, Part 1), and so form a balanced system.

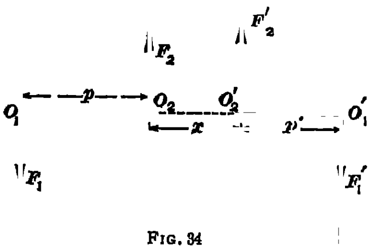


FIG. 34

It follows from this that a couple may be replaced by another couple in the same plane, if the two couples have the same moment and their forces are all parallel, and that the effect of a couple is not altered by moving the couple in its plane parallel to itself.

Case II.—*When the two couples are in the same plane and the lines of action of the forces of one intersect the lines of action of the forces of the other.*

Let the couples be (F_1, p, F_2) and (F_1', p', F_2'), Fig 35. According to the preceding demonstration, the couple (F_1', p', F_2') may be replaced by another couple with one of its forces acting through O_1, provided that the direction of the forces is not changed, and, moreover, the magnitude of the forces may be changed, provided that the arm is so changed as to keep the moment constant Thus, if P_1 is made parallel to F_1' and F_2', and equal to F_1, and if, at the end of the arm $O_1 O_2'' = p$, the force P_2 is applied, equal in magnitude to P_1, but opposite in direction, then the two couples (F_1', p', F_2') and (P_1, O_1O_2'', P_2) will be equivalent, for, by hypothesis, $F_1'p'$ is numerically equal to $F_1 p$, and therefore to $P_1 \times O_1 O_2''$ By transferring the point of application of P_2 to the intersection I of the lines

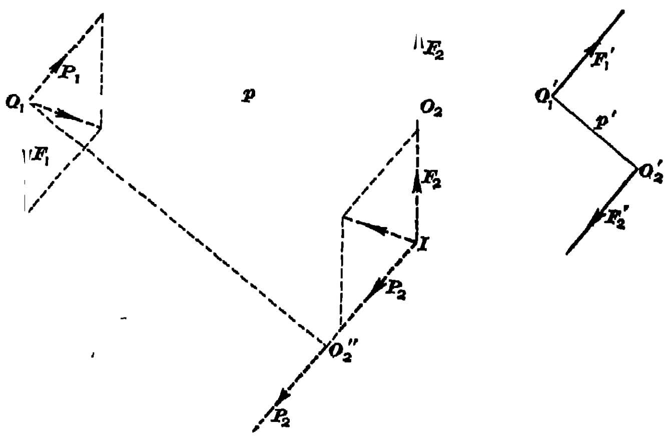


FIG 35

of action of P_2 and F_2, and the point of application of F_2 to the same point, the two couples (F_1, p, F_2) and (F_1', p', F_2') are replaced by the forces P_1 and F_1 acting at O_1, and P_2 and F_2 acting at I Since F_1 and P_1 are equal, their resultant must act along the bisector of the angle $P_1 O_1 F_1$, which evidently is the same as the bisector of $O_2 O_1 O_2''$ Similarly, the resultant of P_2 and F_2 must act along the bisector of $O_2 I O_2''$, and, as P_2 and F_2 are equal and parallel, respectively, to P_1 and F_1, the resultants of the two pair of forces are numerically equal Now, owing to the equality of $O_1 O_2$ and $O_1 O_2''$, the right triangles (not fully shown) $O_1 O_2 I$ and $O_1 O_2'' I$ are equal, and the line $I O_1$ (not shown) is the bisector of both $O_2 O_1 O_2''$ and $O_2 I O_2''$. The two resultants have, therefore, the same line of action, and, as their algebraic sum is zero, they balance each other.

It follows that a couple may be replaced by any other couple acting in the same plane, provided that the two have the same moment

Case III.—*When the planes of the couples do not coincide, but are parallel* (as they must be, since the couples are supposed to be coaxial).

Let PQ and $P'Q'$, Fig 36, be the planes of the two couples, and let (F_1, p_1, F_2) be the couple acting in the plane PQ Since the moment of the couple in the plane $P'Q'$ is $F_1 p_1$, that couple may be replaced by another couple (F_1', p_1', F_2') having the same force and arm as (F_1, p_1, F_2), and the lines of action of whose forces are parallel to the lines of action of F_1 and F_2 This follows from the two cases previously considered, for the couple (F_1', p_1', F_2') is equivalent to any other couple acting in the plane $P'Q'$ and having the same moment. Draw $O_1 O_2'$ and $O_2 O_1'$, meeting at I We may now compound F_1

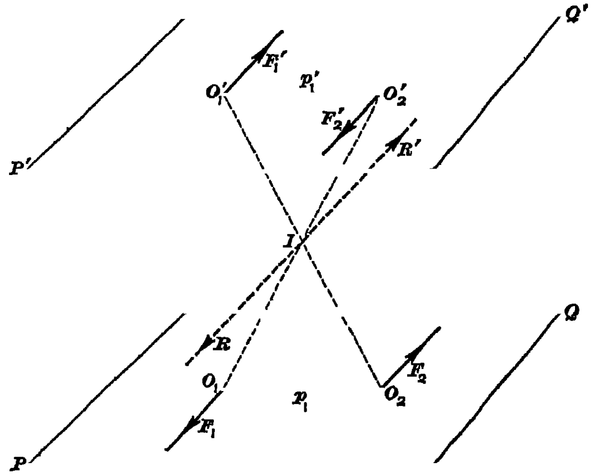


Fig 36

with F_2', and F_2 with F_1'. Since $O_1 O_2$ is equal and parallel to $O_1' O_2'$, the point I is the middle point of $O_1 O_2'$ and $O_2 O_1'$ The resultant of F_1 and F_2' is $R = F_1 + F_2' = 2 F_1$, acting through I The resultant of F_2 and F_1' is $R' = F_2 + F_1' = 2 F_2 = 2 F_1$, acting through I. As these two resultants act in opposite directions along the same line (for they are both parallel to the forces of the couples) and have the same magnitude, they balance each other. Therefore, the two couples to which the two resultants are equivalent balance each other.

47. It follows from the preceding principle that the only
couple are its moment and its axis.
ver, that the axis is not any
ion. Any line perpendicular

to the plane of a couple may be taken as its axis; and, conversely, if the axis of a couple is given, the couple may be supposed to act in any plane perpendicular to that axis. The force and the arm of the couple may be changed at pleasure, provided that their product, which is the moment of the couple, remains unchanged.

48. Resultant and Equilibrant of Any Number of Coaxial Couples.—*The resultant of any number of coaxial couples is a single couple having the same axis as the component couples and whose moment is the algebraic sum of the moments of the component couples.* This principle is a consequence of the one stated in the last article, as can be shown by a process of mathematical reasoning that it is not necessary to give here.

In general, let M_1, M_2, M_3, etc. be the moments of any number of coaxial couples, M_r the moment of their resultant, and M_q the moment of their equilibrant Then,

$$M_r = \Sigma M, \text{ and } M_q = -\Sigma M$$

Also, if several coaxial couples are in equilibrium, the following equation must obtain:

$$M_r = \Sigma M = 0$$

EXAMPLES FOR PRACTICE

1 Find the resultant of the following couples, in which the sign before each parenthesis indicates the direction of the couple (10 lb , 2 ft , 10 lb), − (7 lb , 6 ft , 7 lb.), (25 lb., 12 ft , 25 lb.); − (8 lb , 40 ft., 8 lb) Ans $M_r = -42$ ft -lb.

2 (*a*) Find the lever arm p of a couple with a force of 100 pounds that will balance the following couples: (30 lb , 6 ft , 30 lb.), (20 lb , 5 ft., 20 lb.), − (125 lb , 10 ft , 125 lb); (70 lb , 4 ft , 70 lb), − (80 lb., 3 ft., 80 lb.). (*b*) What is the sign of the balancing couple?

Ans $\begin{cases} (a)\ p = 9.3 \text{ ft.} \\ (b)\ \text{Couple positive} \end{cases}$

3. One of the forces of a couple acts through a certain point O, and is equal to 300 pounds, the moment of the couple is 2,574 foot-pounds. How far from O is the line of action of the other force? Ans 8 58 ft.

4. The moment of a couple is 1,500 foot-pounds. Express it (*a*) as a couple having a force of 75 pounds; (*b*) as a couple having an arm of 12 feet.

Ans $\begin{cases} (a)\ (75 \text{ lb}, 20 \text{ ft}, 75 \text{ lb}) \\ (b)\ (125 \text{ lb.}, 12 \text{ ft.}, 125 \text{ lb.}) \end{cases}$

EQUIVALENCE AND EQUILIBRIUM OF COPLANAR NON-CONCURRENT FORCES

49. To Make the Line of Action of a Force Pass Through a Given Point.—Let F, Fig. 37, be a force acting on a body, and O, a point in the body or rigidly connected with the body. Let the perpendicular distance OP of O from the line of action of the force be denoted by x. It is obvious that, if the two forces F' and $-F'$, equal to each other and to F, are applied at O, they will have no effect on the condition of the body, since they will balance each other. The single force F is, therefore, equivalent to the system of forces $F, F', -F'$, applied as shown. Now, the two forces F and $-F'$ form a couple whose moment is Fx. Therefore, the force F acting along LK is equivalent to an equal and parallel force F' acting through O, together with a couple whose moment is equal to the moment of F about O. In general,

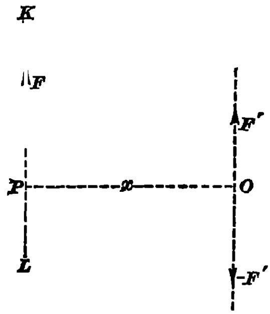


FIG 37

The line of action of a force may be shifted parallel to itself, so that it will pass through any chosen point, provided that a couple is introduced having a moment equal to the moment of the force about that point.

50. Resultant of a Couple and a Force in the Plane of the Couple.—Let the force F, Fig. 38, and a couple (F_1, p, F_2), whose plane contains the line of action of the force, act on a rigid body. The couple may be replaced (Art. 46) by another couple ($F', x, -F'$), whose forces are each numerically equal to F, provided that the lever arm x is such that $F'x = F_1p$, or $x = \frac{F_1p}{F'} = \frac{F_1p}{F}$.

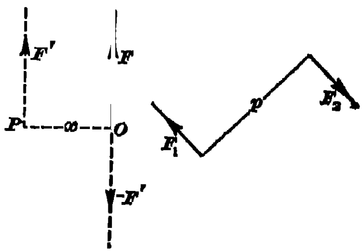


FIG. 38

Also, the couple may be so turned and shifted that one of its forces $-F'$ will act along the line of action of F, but in an opposite direction, as shown. The two forces F and $-F'$ balance each other, so that the system is reduced to the single force F' equal and parallel to F. Therefore,

The resultant of a couple and a single force in the plane of the couple is a single force equal and parallel to the given force, acting along a line whose distance from the line of action of the given force is equal to the moment of the couple divided by the magnitude of the given force.

The direction in which the distance x, or OP, should be measured is indicated by the character of the couple. The moment $F'x$ of the resultant force about any point in the line of action of the given force must have the same sign as the moment of the couple. In the figure, the couple is right-handed, or positive The moment of F' about O must, therefore, be right-handed, which indicates that F' is on the left of F.

51. Resultant of Any Number of Coplanar Non-Concurrent Forces.—Let F_1, F_2, F_3, F_4, Fig. 39, be four non-concurrent forces having their lines of action in one plane, and let O be any point in the plane whose perpendicular distances from the lines of action of the forces are, respectively, p_1, p_2, p_3, p_4. As explained in Art. **49**, F_1 may be replaced by a force F_1', equal and parallel to F_1, acting through O, combined with a couple whose moment is $F_1 p_1$. The other forces may be similarly replaced. The whole system is thus replaced by the four concurrent forces, F_1', F_2', F_3', F_4', acting through O and equal and parallel, respectively to F_1, F_2, F_3, F_4, together with the four couples $F_1 p_1, F_2 p_2, F_3 p_3, F_4 p_4$. The resultant R of the concurrent forces is found as explained elsewhere; the resultant M_r of the couples is a single couple whose moment is the algebraic

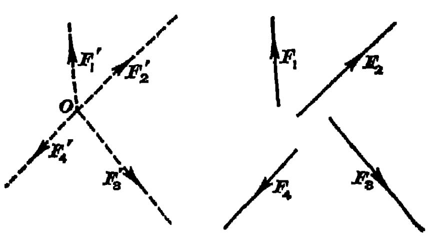


FIG 39

sum of the moments of the forces about O. Finally, the resultant of R and M_r is found as explained in Art. 50.

If the given forces are resolved into components in two directions perpendicular to each other, then, with the usual notation (see *Analytic Statics*, Part 1),

$$X_r = \Sigma X = \Sigma F \cos H \qquad (1)$$
$$Y_r = \Sigma Y = \Sigma F \sin H \qquad (2)$$
$$R = \sqrt{X_r^2 + Y_r^2} \qquad (3)$$
$$\tan H_r = \frac{Y_r}{X_r} = \frac{\Sigma Y}{\Sigma X} \qquad (4)$$

And also, $$M_r = \Sigma F p \qquad (5)$$

52. Conditions of Equilibrium.—When the forces are in equilibrium, both R and M_r must be zero. The resolutes of R must, therefore, be zero, and the algebraic conditions of equilibrium are

$$\Sigma X = \Sigma F \cos H = 0 \qquad (1)$$
$$\Sigma Y = \Sigma F \sin H = 0 \qquad (2)$$
$$M_r = \Sigma M = \Sigma F p = 0 \qquad (3)$$

These conditions may be stated in words thus:

1. *The algebraic sum of the resolutes of the forces in each of any two directions at right angles to each other must be zero.*

2. *The algebraic sum of the moments of the forces about any and every point in their plane must be zero.*

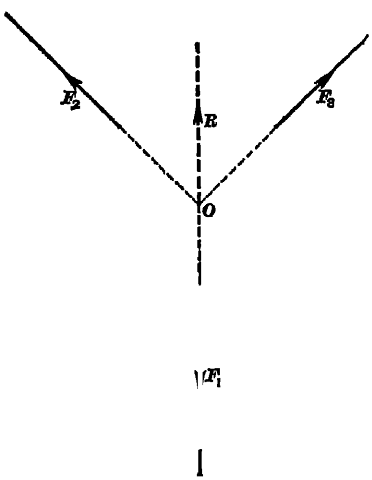


FIG. 40

53. Equilibrium of Three Forces.—If three forces F_1, F_2, F_3, Fig. 40, are in equilibrium, any one of them is the equilibrant of the other two, and, therefore, equal and opposite to their resultant. Thus, F_1 must be equal and opposite to the resultant R of F_2 and F_3. As R passes through the point O of

intersection of the lines of action of F_2 and F_3 and F_1 must balance R, the line of action of F_1 must pass through O. In general,

If three coplanar forces are in equilibrium, they must be concurrent (unless they are parallel), and if the point of intersection of the lines of action of two of the forces is known, the line of action of the other must pass through that point.

EXAMPLE 1.—Four forces, $F_1 = 100$ pounds, $F_2 = 200$ pounds, $F_3 = 125$ pounds, $F_4 = 150$ pounds, Fig. 41, act on a horizontal

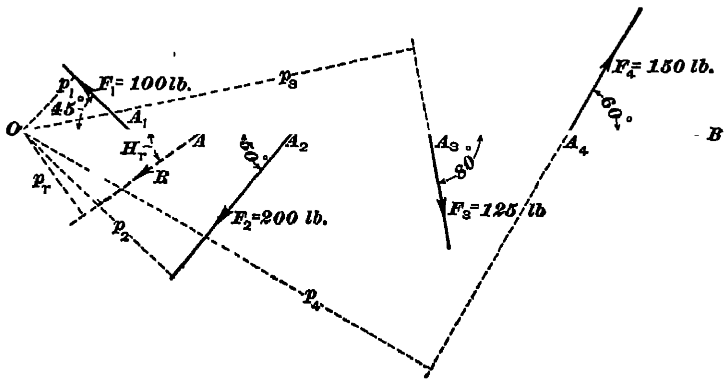


FIG. 41

lever OB. The inclinations of their lines of action to the horizontal are as shown, and $OA_1 = 4$ feet, $OA_2 = 10$ feet, $OA_3 = 15$ feet, $OA_4 = 20$ feet. Required the magnitude, direction, and line of action of the resultant R.

SOLUTION.—Using formulas **1** and **2** of Art. **51**, we have, since $F_1 = 100$, $F_2 = 200$, $F_3 = 125$, $F_4 = 150$, and $H_1 = 45°$, $H_2 = 50°$, $H_3 = 80°$, $H_4 = 60°$,

$$X_r = \Sigma F \cos H = -F_1 \cos H_1 - F_2 \cos H_2 + F_3 \cos H_3 + F_4 \cos H_4$$
$$= -100 \cos 45° - 200 \cos 50° + 125 \cos 80° + 150 \cos 60° = -102.56$$
$$Y_r = \Sigma F \sin H = F_1 \sin H_1 - F_2 \sin H_2 - F_3 \sin H_3 + F_4 \sin H_4$$
$$= 100 \sin 45° - 200 \sin 50° - 125 \sin 80° + 150 \sin 60° = -75.694$$

By formula **3** of Art. **51**,

$$R = \sqrt{X_r^2 + Y_r^2} = \sqrt{102.56^2 + 75.694^2} = 127.47 \text{ lb} \quad \text{Ans}$$

By formula **4** of Art. **51**,

$$\tan H_r = \frac{Y_r}{X_r} = \frac{75.694}{102.56}; \text{ whence } H_r = 36° \ 25' \ 40''. \quad \text{Ans.}$$

Since $p_1 = OA_1 \sin 45° = 4 \sin 45° = 2.8284$; $p_2 = OA_2 \sin 50° = 10 \sin 50° = 7.6604$, $p_3 = OA_3 \sin 80° = 15 \sin 80° = 14.772$, and

$p_4 = OA_4 \sin 60° = 20 \sin 60° = 17\ 321$, formula **5** of Art **51** gives

$$M_r = \Sigma F p = -F_1 p_1 + F_2 p_2 + F_3 p_3 - F_4 p_4$$
$$= -100 \times 2\ 8284 + 200 \times 7\ 6604 + 125 \times 14\ 772 - 150 \times 17\ 321$$
$$= 497\ 59 \text{ ft -lb}$$

The value of Y_r shows that the force R acts downwards, its line of action must, therefore, be on the right of O, since its moment about this point is positive Hence,

$$M_r = R p_r, \text{ whence } p_r = \frac{M_r}{R} = \frac{497\ 59}{127\ 47} = 3\ 9035$$

Also, $$OA = \frac{p_r}{\sin H_r} = \frac{3\ 9035}{\sin 36° 25' 40''} = 6\ 5739 \quad \text{Ans}$$

EXAMPLE 2 —Two forces, $F_1 = 2{,}000$ pounds and $F_2 = 800$ pounds, Fig 42, act on a horizontal beam 25 feet long resting on two supports

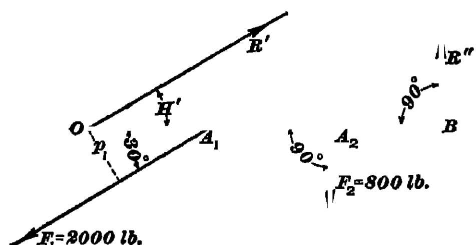


FIG 42

at its extremities O and B The distances OA_1 and OA_2 are, respectively, 8 and 16 feet. The inclinations of F_1 and F_2 are as shown. It is known that the reaction R'' at B is vertical Required the magnitudes of the two reactions R'' and R', and the inclination H' of R' to the horizontal

SOLUTION —Since the moment of R' about O is zero, R'' may be found by taking moments about that point and using formula **3** of Art **52**, which gives,

$$F_1 p_1 + F_2 \times OA_2 - R'' \times OB = 0;$$

that is, $$2{,}000 \times 8 \sin 30° + 800 \times 16 - R'' \times 25 = 0;$$

whence, $$R'' = \frac{2{,}000 \times 8 \sin 30° + 800 \times 16}{25} = 832 \text{ lb. Ans.}$$

By formula **1** of Art. **52**, $\Sigma F \cos H = 0$; that is,

$$R' \cos H' - F_1 \cos 30° = 0;$$

whence,

$$R' \cos H' = F_1 \cos 30° = 2{,}000 \cos 30° = 1{,}732 \qquad (1)$$

Similarly, by formula **2** of Art. **52**, $\Sigma F \sin H = 0$; that is,

$$R' \sin H' - F_1 \sin 30° - F_2 + R'' = 0;$$

whence, $$R' \sin H' = F_1 \sin 30° + F_2 - R''$$
$$= 2{,}000 \sin 30° + 800 - 832 = 968 \qquad (2)$$

Dividing (2) by (1),

$$\frac{R' \sin H'}{R' \cos H'} = \tan H' = \frac{968}{1{,}732};$$

whence, $H' = 29° 12'$, nearly Ans.

From (2), $R' = \frac{968}{\sin H'} = 1{,}984$ lb. Ans.

APPLICATIONS

54. Mutual Reactions.—In Fig. 43 (*a*) are represented two bars hinged at A_1 and A_2, and to each other at B. The bars are acted on by forces F_1 and F_2. Through the joint B, the bar A_1B exerts on A_2B a force R_2', which is equal and opposite to the force R_1' exerted by A_2B on A_1B. This force $R_1' = R_2'$ is the **mutual reaction** between the two bars at the joint B. How it is determined will be explained presently. Since the system formed by the two bars is in equilibrium under the action of the external forces F_1, F_2, R_1, R_2, these forces form a balanced system to which the general equations of equilibrium can be applied. From these equations, the reactions R_1 and R_2 can be found when all other conditions, such as distances, etc., are known.

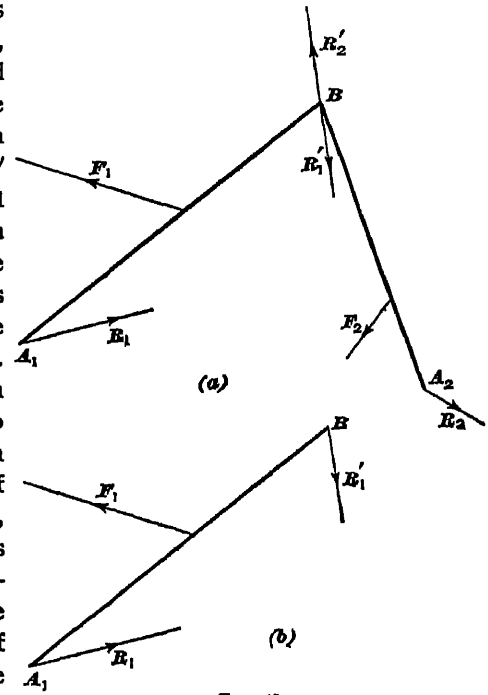


FIG. 43

The mutual reaction at B is determined by applying the principle of separate equilibrium (see *Analytical Statics*,

Part 1). The part A_2B may be removed, and A_1B treated as a free or separate body, provided that a force is introduced at B equal to the force exerted by A_2B on A_1B, that is, a force equal to R_1'. This condition is represented in Fig 43 (b), where A_1B is shown as a free body acted on by the external forces R_1, F_1, and R_1'. By applying to the system constituted by these forces the general equations of equilibrium, R_1' may be determined.

55. Method of Sections. The free-body principle finds a very useful application in the determination of stresses in framed structures by the **method of sections**, illustrated in Fig. 44. The truss $A_1B_1B_2A_2$ rests on piers A_1 and A_2, and is loaded at the joints C_1, C, and C_2, as shown.

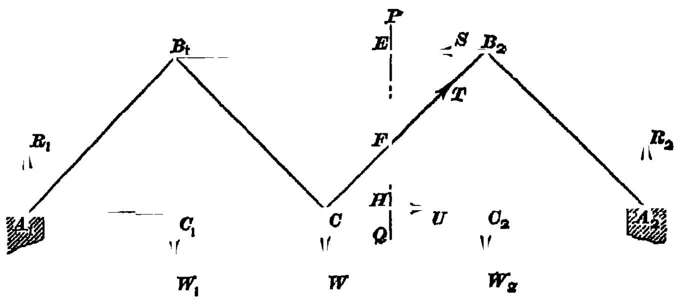


FIG. 44

Let the reactions R_1 and R_2 first be determined. Considering the truss as a whole, the external forces acting on it are the weights W_1, W, and W_2, and the reactions R_1 and R_2. These forces form a balanced system, and, therefore, the algebraic sum of their moments about any point must be zero (Art. 52). Taking moments about A_1, in order to eliminate R_1,

$$W_2 \times A_1C_2 + W \times A_1C + W_1 \times A_1C_1 - R_2 \times A_1A_2 = 0,$$

from which R_2 can be found. To find R_1, moments may be taken about A_2; or, knowing R_2, R_1 may be found from the equation $\Sigma Y = 0$, which in this case gives

$$R_1 + R_2 - W_1 - W_2 - W = 0$$

The stresses in the members may be found by the method explained in *Analytic Statics*, Part 1, proceeding joint by joint,

beginning either at A_1 or at A_2, since now R_1 and R_2 are known. Or the method of sections, referred to above, may be used, as follows:

Let it be required to find the stresses in the member CC_2. Imagine the truss to be cut in two by a plane PQ intersecting CC_2 and the members CB_2 and B_1B_2, which meet at B_2.

The part of the truss at the right of PQ may be supposed to be removed by introducing at the points of separation E, F, H, forces S, T, U, equal to the actions of B_2E on EB_1, B_2F on FC, and C_2H on HC, respectively, which are the measures of the stresses in the three members intersected by the plane The part A_1B_1EFH may now be treated as a free body acted on by the external forces R_1, W_1, W, S, T, and U, of which only the last three are unknown. If moments are taken about B_2 (the point of intersection of S and T), the forces S and T will be eliminated, and an equation will be obtained in which the only unknown quantity will be U.

The same result would have been obtained if the members CC_2, B_2C_2, and B_2A_2 had been cut by the plane PQ.

To find S, moments are taken about C. In every case, several members should be cut, of which all but one meet at a joint, and this joint should be taken as the origin of moments. If, however, the stresses in some of the members cut are known, it is immaterial whether they meet at a joint or not: the only thing necessary is that, of the members whose stresses are unknown, all but one should be concurrent.

FRICTION

SLIDING FRICTION

DEFINITIONS AND GENERAL PRINCIPLES

56. Definition of Sliding Friction.—Let a block CDE, Fig. 45, rest on a horizontal surface AB capable of resisting or balancing the weight W of the block. The block will then be in equilibrium under the action of the weight W and the reaction R, the latter being equal and opposite to W. If, now, a horizontal force F (which, for convenience, will be supposed to act along a line passing through the c. g of the block) is applied, and it is assumed that no other force than W, R, and F acts on the block, the latter being under the action of an unbalanced force, will move in the direction of F with an acceleration equal to $\frac{F}{W} g$ (see *Fundamental Principles of Mechanics*). So long as there is no other force acting on the block, motion will ensue, however small F may be.

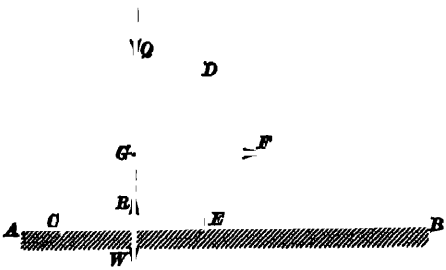


FIG 45

Experience, however, shows that a small force, whether horizontal or not, often produces no effect on a body resting on a surface, and sometimes a very great force is required before the equilibrium of the body is disturbed. Thus, to drag a trunk or a box over the floor may require the efforts of several strong men. We also know that the resistance is

greater the rougher the surfaces in contact: a rough box is not so easily dragged over the sidewalk as a polished stone block over a smooth wooden floor.

There is, then, a force brought into action whenever there is a tendency of a body to slide on another. This force, whose effect is to prevent, or which tends to prevent, motion, is called **sliding friction**, or simply **friction**. It is obviously caused by the roughness of the surfaces in contact. No matter how smooth a surface may appear, it always has small projections or elevations separated by small depressions. When the surfaces of two bodies are in contact, the projections of one surface go into the hollows of the other, the two surfaces thus become more or less interlocked and cannot slide freely on each other. In order to cause sliding, a force is necessary, whose magnitude depends on the roughness of the two surfaces. As this roughness varies with different bodies, it may be anticipated (and this is known from experience to be the case) that friction must be a very variable force, depending both on the nature and on the conditions of the bodies in contact.

57. Limiting Equilibrium.—Given the block CDE, Fig. 45, resting on the surface AB, the force F may either move it or leave its equilibrium undisturbed. In the former case, F must be greater than the friction; in the latter case, F must be less than the friction Let F_m be the greatest force that can be applied to the block without moving it. Then, a force equal and opposite to F_m will represent the maximum friction that can exist between the block and the surface AB, any force greater than F_m will cause motion, and any force less than F_m will be balanced by the friction. But it is not to be supposed that the friction is constantly equal to F_m: so long as the applied force is less than F_m, the friction is just equal and opposite to the applied force; the friction grows with the applied force up to the value F_m, beyond which equilibrium ceases to exist and the body moves under the action of the difference between the applied force and the maximum friction F_m.

When the block is acted on by a force equal to F_m, it is said to be in a condition of **limiting equilibrium**, or on **the point of moving**, for the least increase in the applied force is sufficient to produce motion. In this case, there exists between the two bodies AB and CDE the greatest possible friction that under the given circumstances can exist between them. This maximum force of friction is called **limiting friction**, and will hereafter be designated by P_m. Numerically, $P_m = F_m$.

58. Passive Forces.—Friction, like many other resistances, is a **passive force**—that is, a force preventing motion, but not producing it. The reason for this is that friction is brought into action by the application of other forces to which it is opposed; and, as it can never exceed those forces, it can never produce motion in the direction of its own line of action. The same is true of the reactions of supports. A pier may be capable of exerting a reaction of 1,000 tons; but, if a stone weighing 1 pound is placed on it, the pier will exert on the stone an upward pressure of only 1 pound, or just enough to balance the weight of the stone.

Forces producing, or tending to produce, motion, are called **active forces**.

Passive forces are balancing forces and never acquire greater magnitudes than the active forces they oppose. Usually, as in the case of friction and reactions, a passive force cannot exceed a certain limit; but between zero and that limit it can have any value, and so long as the active force to which the passive force is opposed does not exceed that limit there will be equilibrium.

59. Shifting of the Line of Action of the Reaction. The force of friction is a **tangential force**, by which is meant a force acting along the surface of contact of the two bodies between which it is exerted. In Fig. 45, the force of friction is not directly opposed to F, but acts along the surface CE, in the direction EC. How, then, can this force balance F, not being in line with it?

The reason is that, on the application of F, the reaction R is shifted so that it no longer passes through the point where the vertical through the c. g. of the body meets the supporting surface. This is illustrated in Fig. 46. The line of action of the applied force F meets the vertical line through G at O. The resultant of W and F, found in the usual manner, is F_r, whose line of action meets AB at O'. Transferring F_r to O' (where, for convenience, it is represented by F_r'), and again resolving it into its components $F' = F$ and $W' = W$, it is seen that, in order that there may be equilibrium, the reaction R must be equal and

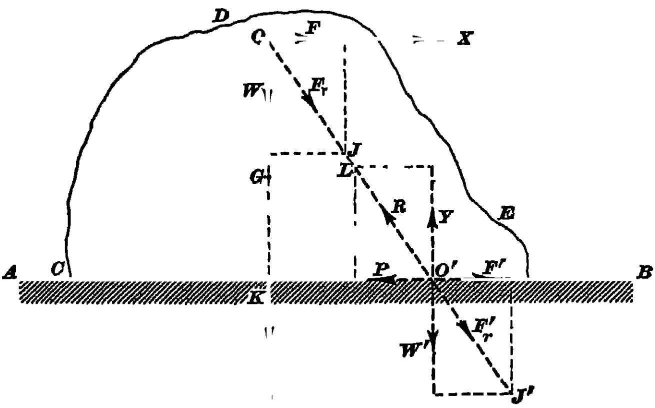


Fig. 46

opposite to F_r', and its components Y and P must be, respectively, equal and opposite to $W'(=W)$ and $F'(=F)$. The vertical component Y is the resistance of AB to direct or normal pressure; and at present it will be assumed that the surfaces in contact are capable of offering this resistance, whatever the value of Y or W may be. The horizontal component P is the force of friction acting along the surface of contact. In the case illustrated in the figure, F is supposed less than F_m—that is, less than the maximum force that can be opposed by the friction. Therefore, P, which can take any value between 0 and F_m, will, in this case, be just equal to F', or F, and balance the latter force.

It thus appears that the effect of the friction is to shift the point of application of the reaction from K to O', and to change the line of action of that reaction from the vertical direction to the direction $O'L$.

The body CDE may be considered as being acted on by the two equal and opposite forces R and F_r, or by the two couples $-(W, KO', Y)$ and (F, OK, P). Let the student show that the moments of these two couples are numerically equal.

60. Case in Which F is Greater Than P_m.—We shall now consider the case in which F is greater than F_m, or P_m. In Fig. 47, the block CDE is acted on by its own

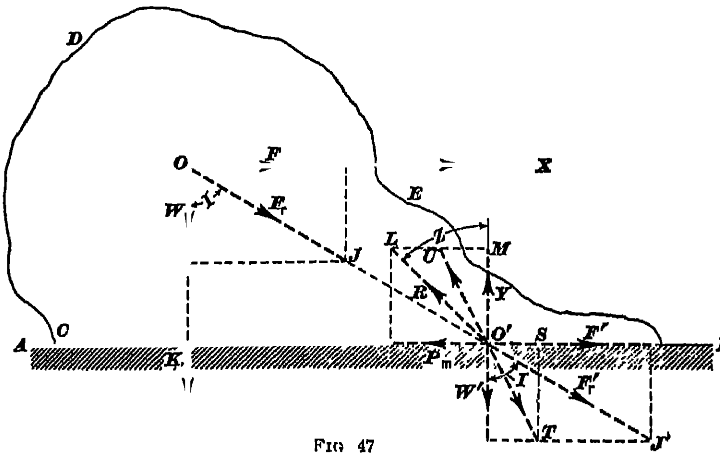


Fig 47

weight W and the horizontal force F. As before, $F_r' = F_r$ is the resultant of W and F, transferred to the point O' on the surface of contact; its components are $F' = F$ and $W' = W$. As, in this case, F is greater than F_m, the force of friction, which has its maximum value $P_m = F_m$, is not sufficient to balance F', and there will be sliding of the block on AB. The reaction R of the latter surface is the resultant of $Y = W' = W$ and P_m. This reaction is evidently less than F_r', and its line of action makes with the normal $O'M$ to AB an angle Z less than the angle I made by the line of action of F_r with that normal.

If, Y (that is, W or W') remaining constant, F is made less than P_m, say equal to $O'S$, the resultant will be $O'T$, and the reaction will be $O'U$, whose components are $Y = W$ and MU equal to the friction P, which in this case, will be equal to $O'S$.

The conditions of equilibrium may, therefore, be stated by saying either that F must not be greater than P_m, or that I must not be greater than Z.

ANGLE AND COEFFICIENT OF FRICTION

61. Maximum Resistance.—The results of the foregoing discussion may now be generalized. Let N, Fig. 48, be the normal force between two bodies whose surface of contact is AB. The force N is the normal component of the resultant force acting on CDE (or on the other body), when that resultant, after its point of application has been trans-

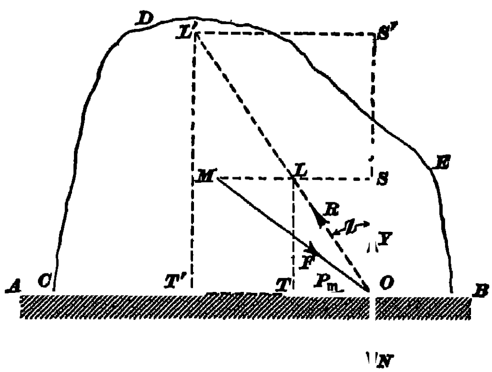


FIG. 48

ferred to the intersection of its line of action with the surface of contact, is resolved into two components, one parallel, and one normal, to that surface. Given, besides the normal component N (usually called the **normal pressure**), all other conditions—such as the nature of the two bodies, the extent of their surface of contact, etc.—the maximum

friction P_m that can exist between them is determined by actual experiment Once P_m is known, the maximum reaction R, which we shall call the **maximum resistance** that can exist between the two surfaces, is

$$R = \sqrt{Y^2 + P_m^2} = \sqrt{N^2 + P_m^2}$$

The angle made by the line of action of the reaction with the normal OS to the surface of contact is given by the familiar expression,

$$\tan Z = \frac{LS}{OS} = \frac{P_m}{Y} = \frac{P_m}{N}$$

If, the normal pressure and all other conditions remaining constant, a force F greater than R is applied to CDE (in which case N is the normal component of F), it is obvious that there cannot be equilibrium, since the component of F parallel to AB, which component is equal to MS, is greater than P_m. If, on the contrary, F is less than R, or equal to R—that is, if its line of action falls within the angle Z or coincides with LO—equilibrium will obtain.

62. Angle of Friction: Condition of Equilibrium. The angle Z, Fig 48, is called the **angle of friction**, and may be defined as the angle between the line of action of the maximum resistance and the normal to the surface of contact.

The condition of equilibrium explained in the preceding articles may be thus stated:

In order that there may not be sliding between two bodies in contact, it is necessary and sufficient that the resultant of the applied forces (as F_r, Fig. 47) shall not make with the normal to the surface of contact an angle greater than the angle of friction.

63. Coefficient of Friction.—Until recently, it was thought that, for any two given substances, the maximum friction P_m was directly proportional to the normal pressure N and independent of all other circumstances. According to this view, if P_m', P_m'', P_m''', etc. are the maximum frictions corresponding to the normal pressures N', N'', N''', etc., then $\frac{P_m'}{N'} = \frac{P_m''}{N''} = \frac{P_m'''}{N'''}$. The common value of these ratios will be denoted by c.

Having determined the ratio c of the friction to the normal pressure for any particular case, the friction in any other case could be at once found from the relation $\frac{P_m}{N} = c$; whence, $P_m = cN$. Also, since

$$\tan Z = \frac{P_m}{N} = c$$

it would follow that the angle of friction was constant for every two substances sliding on each other.

That there is generally some dependence of the force of friction on the pressure is a familiar fact. Thus, referring again to Fig. 45, daily experience shows that, if a pressure Q is applied to the block, the effort required to drag the block will increase as the pressure Q increases. The relation between pressure and friction, however, is not always so simple as stated above.

The law that *friction is proportional to pressure* is approximately true only in some cases, or under certain conditions. What these conditions are, and how friction varies under different conditions, are problems to be solved by direct experiment.

Although the ratio of P_m to N is usually variable, it is customary and convenient to express P_m as a fraction of the normal pressure, and write,

$$P_m = fN \qquad (1)$$

The factor f, or the ratio of the maximum friction to the normal pressure, is called the **coefficient of friction, or friction factor.** It has been determined experimentally for various substances under various circumstances; its values have been tabulated, and the approximate laws of its variations stated. Here, the theory only of friction will be considered. In order, however, that a general idea of the values of f may be obtained, it will be remarked that, in the majority of cases coming within the practice of the mechanical engineer, f is usually much less than .5. Thus, for unlubricated metals sliding on one another, the average value of f is about .18; for dry and smooth wood sliding on the same, f averages about .38; and for wood sliding on metal, both

dry and smooth, about .4. In civil-engineering work, however, values of f often occur that exceed .5, as in the case of brick sliding on brick or on stone, masonry on brickwork, etc.

Since $\tan Z = \frac{P_m}{N}$, which is the same as the value of f from formula 1, we may write

$$\tan Z = f \qquad (2)$$

The angle of friction may, therefore, be defined as an angle whose trigonometric tangent is equal to the coefficient of friction, it being understood that one of the sides of this angle is the normal to the surface of contact of the two bodies whose friction is considered.

64. Static Friction and Kinetic Friction. In order to set a body in motion over another body, a force is necessary whose component parallel to the surface of contact is greater than P_m. Once the body is in motion, it seems that it should continue in motion, by virtue of its inertia, if a force just equal to P_m were constantly applied to it; for this force would be sufficient to overcome the friction. Experience, however, shows that the force necessary to start a body sliding on another is almost always greater than the force necessary to keep it sliding, once motion has begun. In other words, the resistance of friction is greater when motion is to be produced, than when it is to be maintained. In the former case, the friction is called **friction of rest**, or **static friction**; in the latter case, **friction of motion**, **dynamic friction**, or **kinetic friction**. In either case, the maximum friction will be here designated by P_m, and the coefficient of friction by f, it being understood that, as a rule, P_m and f have different values for the two cases.

65. In practice, it is necessary to bear in mind what the function of friction is when any particular problem is to be solved. In designing a machine, where motion has to be produced and maintained, sufficient force should be allowed to overcome the friction of rest, as otherwise the machine could not be started. But, in calculating the **efficiency** (a term to be defined elsewhere) of the machine while in

motion, the friction of motion should be used. In providing for the equilibrium of a structure, however, where friction is a favorable force, the friction of motion should be used; for, although under ordinary circumstances the structure, being in a state of rest, will offer a frictional resistance equal to the maximum friction of rest, the least shock is often sufficient to produce a disturbance of equilibrium; the structure is, so to speak, started, and then the only resistance preventing it from continuing to move will be the friction of motion. This is especially the case in structures subjected to shocks, such as bridges and engine foundations.

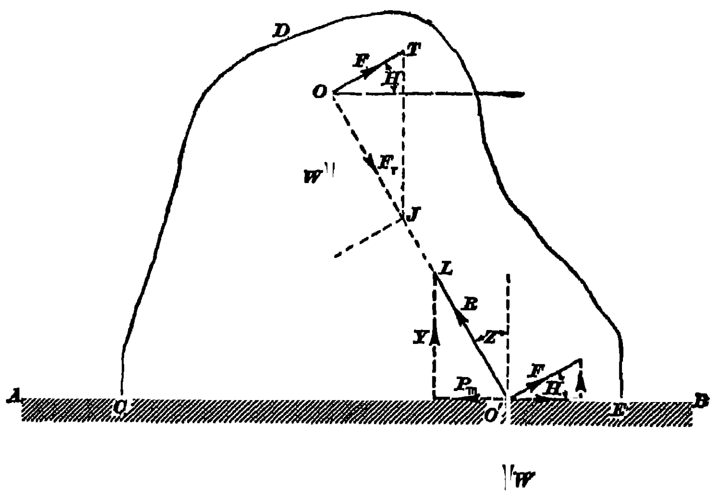


FIG. 49

EXAMPLE —A block CDE, Fig 49, whose weight is W, rests on a horizontal surface AB The coefficient of friction between the block and the surface is f. A force F, acting in a vertical plane containing the c g of the block, is applied at an angle H to the horizontal. Required the magnitude of F, that the block may be on the point of sliding along AB.

SOLUTION —All problems similar to this may be solved in two manners, and the result found directly in terms of either f or Z. Of

course, one result can be transformed algebraically into the other from the relation $\tan Z = f$

1 Let O be the intersection of the line of action of F with the vertical through the c g of the block The latter is held in equilibrium by the forces F, W, and the total reaction R of AB, whose line of action $O'O$ must pass through O (Art 53) As explained in Art 62, this line of action must make with the normal to AB, or with the vertical, an angle equal to Z Therefore, F_r being equal to and collinear with R, angle $OJT = WOJ = Z$ The triangle OJT gives

$$F = \frac{TJ}{\sin TOJ} \sin OJT = \frac{W}{\sin(WOT - WOJ)} \sin Z$$

$$= \frac{W}{\sin(90° + H - WOJ)} \sin Z$$

$$= \frac{W}{\sin(90° + H - Z)} \sin Z = \frac{W}{\cos(H - Z)} \sin Z \qquad (1)$$

2 The line of action of the resultant F_r of the forces F and W meets the surface AB at O'. Let F and W be transferred to this point, as shown. Resolving F into its horizontal and vertical components $F \cos H$ and $F \sin H$, the resultant normal pressure acting at O' is

$$N = W - F \sin H$$

Therefore, the resistance of friction is:

$$P_m = Nf = (W - F \sin H) f$$

As this force must balance the horizontal component of F, there results:

$$(W - F \sin H) f = F \cos H;$$

whence,

$$F = \frac{Wf}{\cos H + f \sin H} \qquad (2)$$

To reduce equation (2) to equation (1), we have

$$\frac{Wf}{\cos H + f \sin H} = W \frac{\tan Z}{\cos H + \tan Z \sin H}$$

$$= W \frac{\frac{\sin Z}{\cos Z}}{\cos H + \frac{\sin Z}{\cos Z} \sin H} = W \frac{\sin Z}{\cos H \cos Z + \sin H \sin Z}$$

$$= W \frac{\sin Z}{\cos(H - Z)}$$

EXAMPLES FOR PRACTICE

1. Find the angle of friction, to the nearest minute, corresponding to each of the following coefficients of friction: (*a*) $f = .15$; (*b*) $f = .25$; (*c*) $f = .50$, (*d*) $f = .65$.

$$\text{Ans.} \begin{cases} (a) & Z = 8° 32' \\ (b) & Z = 14° 2' \\ (c) & Z = 26° 34' \\ (d) & Z = 33° 1' \end{cases}$$

2. Find the coefficient of friction corresponding to each of the following angles of friction (*a*) $Z = 12° 15'$, (*b*) $Z = 30°$, (*c*) $Z = 8° 35'$; (*d*) $Z = 3° 17'$.

$$\text{Ans} \begin{cases} (a)\ f = .217 \\ (b)\ f = 577 \\ (c)\ f = .151 \\ (d)\ f = 057 \end{cases}$$

3 A block of marble weighing 100 pounds is kept sliding with uniform velocity on a horizontal pine floor by a force inclined to the horizontal at an angle $H = 30°$. If the coefficient of kinetic friction between marble and pine is 45, what must the magnitude of the force be (*a*) if H is an angle of depression? (*b*) if H is an angle of elevation?

$$\text{Ans} \begin{cases} (a)\ F = 70\ 22 \text{ lb.} \\ (b)\ F = 41\ 25 \text{ lb.} \end{cases}$$

4. A force of 15 pounds, inclined to the horizontal at an angle of elevation of 30°, is just enough to keep a block of cast iron, weighing 100 pounds, sliding uniformly on a horizontal cast-iron plate Find the coefficient and the angle of friction

$$\text{Ans} \begin{cases} f = 14 \\ Z = 7° 58' \end{cases}$$

5. Taking the angle of friction of rest for brick sliding on brick as 35° 30', with what normal pressure N must a brick weighing 6 pounds be pressed against a vertical brick wall that the brick may not slide down? (Use only two decimal places for f.)

Ans $N = 8\ 45$ lb.

RESISTANCE TO ROLLING

66. Cause of Resistance to Rolling.—Let a homogeneous cylinder AD, Fig. 50, of radius r, rest on a horizontal surface $X'X$. If a horizontal force F is gradually applied to the cylinder along any line NL perpendicular to the axis of the cylinder, it will be found that no motion can be produced before the force F exceeds a certain limit. Now, the line of action of the weight W of the cylinder meets the supporting surface at A, directly under the center O. Did the cylinder touch the surface $X'X$ only at A, it is obvious that any horizontal force, however small, would cause motion; for, as the resultant of that force and the weight could not be vertical, its line of action could not pass through A, and, therefore, such a resultant could not be balanced by the reaction of the supporting surface. And, since experiment shows that it is possible to apply a horizontal force to the cylinder without causing motion, it follows

that not only the point A, but a part AB of the cylinder must be in contact with $X'X$; in which case it is easy to understand how the resultant F_r of F and W may be counteracted by the reaction R of the surface $X'X$. That contact cannot take place at the point A only is otherwise evident from the fact that all substances are more or less compressible, so that, while the weight of the cylinder causes a small depression in the supporting surface, the reaction of the latter causes a small flattening of the cylindrical surface, as shown in the figure.

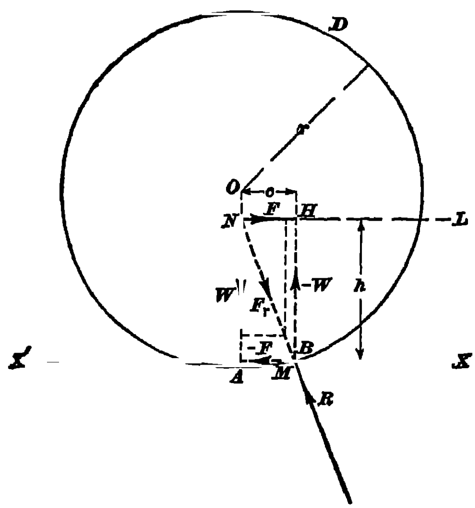


FIG 50

67. Coefficient of Rolling Friction.—Suppose that the cylinder is in a state of limiting equilibrium with respect to rolling; that is, that any increase of the force F will cause the cylinder to roll. Let the line of action of the resultant of W and F meet the surface of contact at M, at a horizontal distance c from the theoretical point of contact A. Resolving the reaction R into its components $-F$ and $-W$, it will be seen that the cylinder is in equilibrium under the action of two couples; namely, $(F, MH, -F)$, and $-(W, NH, -W)$. Therefore,

$$F \times MH = W \times NH = Wc$$

But, as the deformation of the surfaces of contact is very small, we may write, with sufficient approximation, $MH = AN = h$; and, therefore, $Fh = Wc$; whence

$$F = W\frac{c}{h}$$

It has been ascertained by experiment that the distance c is independent of the dimensions and weight of the cylinder and depends only on the materials of the two surfaces in contact The particular value of c for any two materials rolling on each other (it is not necessary that one of the surfaces should be a plane, as $X'X$) is called the **coefficient of rolling friction** for those two materials This coefficient is not an abstract number, but a length, and its value, of course, depends on the unit of length used. The following are approximate values of the coefficient c.

For elm rolling on oak, $c = .032$ inch

For iron on iron and steel on steel, $c = .02$ inch.

It appears from the foregoing formula that, for any two materials, the magnitude of the force F producing limiting equilibrium depends on its lever arm h. What is necessary and sufficient in order to produce limiting equilibrium is that the moment Fh of the applied force should balance the constant moment Wc The resistance to rolling may, therefore, be said to be expressed by a couple Wc rather than by a single force.

This constant couple, whose moment is obtained *by multiplying the normal pressure acting between the two surfaces by the coefficient of rolling friction*, is called a **friction couple.**

68. To Determine Whether Equilibrium Will Be Broken by Sliding or by Rolling.—Referring again to Fig. 50, it must be noticed that rolling about M (practically about A) cannot occur if the cylinder slides before Fh reaches the limit Wc (here W denotes the sum of all the normal forces acting between the two surfaces) If f is the coefficient of sliding friction, F must not be greater than Wf, or $W\frac{c}{h}$ must not be greater than Wf; therefore, $\frac{c}{h}$ must not be greater than f, or c must not be greater than fh.

If c is greater than fh, sliding will begin before rolling can take place. If $c = fh$, sliding and rolling will begin simultaneously.

THE INCLINED PLANE

69. Definitions.—An **inclined plane** is, as its name implies, a plane surface inclined to the horizon. In Fig. 51, the plane $PQRS$, making with the horizontal plane $PTUS$ (or any other horizontal plane) an angle QPT, is an inclined plane.

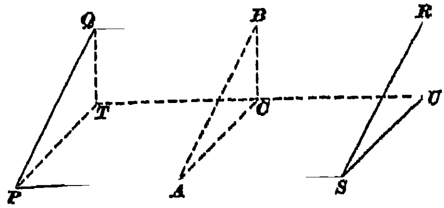


Fig. 51

70. Any vertical plane perpendicular to an inclined plane is called a **principal plane**, and its intersection with the inclined plane is called a **line of declivity**. A line of declivity may also be defined as a line lying in the inclined plane and perpendicular to the intersection of the latter plane with any horizontal plane. In Fig. 51, PQT, ABC, SRU, being vertical planes perpendicular to $PQRS$, are principal planes, and the lines QP, BA, RS are lines of declivity.

71. The angle between an inclined plane and the horizontal is called the **angle** of the inclined plane, and is the same as the angle that any line of declivity makes with the horizontal. Thus, in Fig. 51, the angle of the plane is the common value of the angles QPT, BAC, RSU made by the lines of declivity QP, BA, RS with the horizontal.

72. In practice, an inclined plane is always the surface of some body, as a plank, an inclined rail, the side of a hill, etc. If, for any special purposes, it is desirable to take into account a definite extent of this surface, as $PQRS$ in Fig. 51, the line AB (or any other parallel to it and included between PS and QR) is called the **length** of the inclined plane; BC is the **height**, and AC the **base**.

73. Equilibrium of a Body on an Inclined Plane. Let J, MJ', Fig. 52, be a body resting on an inclined plane AB. The view here represented is a section made by a principal plane through the c. g. of the body. All

forces are supposed to lie in that plane. The angle of the inclined plane is $BAC = H$. If only the portion AB were considered, AB would be the length, BC the height, and AC the base. But these dimensions are not needed for the purpose of the present discussion. The line VV' is a vertical through the c. g. of the body, W is the weight of

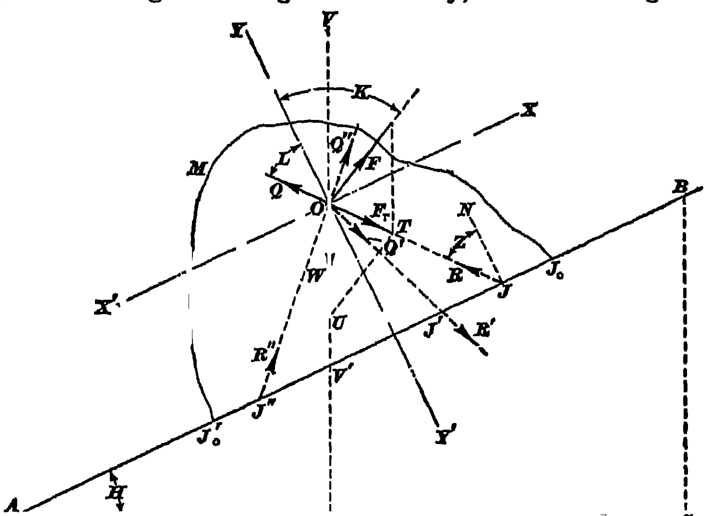


FIG 52

the body, and F a force whose line of action meets VV' at O. Through O draw YY' and XX', the former perpendicular, the latter parallel, to AB. Let the line of action of F make an angle K with YY'. Denote the equilibrant of F and W by Q, their resultant by F_r, and the angle that Q and F_r make with YY' by L. The triangle OTU gives

$$\frac{UT}{OU} = \frac{\sin UOT}{\sin OTU}$$

Now, $UT = F$; $OU = W$; $\sin UOT = \sin (L + H)$; and $\sin OTU = \sin TOF = \sin (180 - QOF) = \sin QOF = \sin (L + K)$. Substituting in the preceding equation,

$$\frac{F}{W} = \frac{\sin (L + H)}{\sin (L + K)},$$

whence, $$F = W \frac{\sin (L + H)}{\sin (L + K)}$$

The equilibrant Q of F and W is the reaction R of the inclined plane, acting through the point J where the line of action of the resultant F_r meets the plane. When the body is in a position of limiting equilibrium with respect to sliding, R makes with the normal JN an angle equal to the angle of friction Z (Art. 62). As JN is parallel to YY', it follows that, in this case, $L = Z$, and the force F is the maximum force acting along the line OF that the body can resist without sliding. It is often said that this is the force that is just enough to start the body moving over the plane; but this is not correct. If the body is already moving, this force, being just enough to balance the friction, will keep the body moving with constant velocity. If the body is at rest, the force will simply keep it in a condition of limiting equilibrium; the body will not move, but the least increase in the force will be sufficient to produce motion.

74. To find the value of F for which the body is in a condition of limiting equilibrium (or will move with constant velocity, once started), it is necessary to distinguish two cases.

Case I.—*The body is on the point of moving up the plane.*

In this case, K must be on the right (in the figure) of YY', but it may be acute, right, or obtuse. If, however, K is acute, it must be greater than H; that is, F must act on the right of $V'V$; for, if F lay on the left of $V'V$, the resultant F_r would act either in the angle VOX' or in the angle $V'OX'$: in the former case, the equilibrant would be some force directed like Q', and, in order that the plane might furnish this equilibrant by its resistance, the reaction R' should have a normal component acting downwards. This cannot take place, as the plane is not supposed to be capable of exerting any downward reaction. If the resultant fell in the angle $V'OX'$, the equilibrant would be a force like Q''; this equilibrant might be furnished by the reaction R'' of the plane; but in this case the friction, being the component of R'' parallel to the plane, would act upwards, and the body could not be on the point of moving upwards.

It being, then, understood that, in the case under consideration, K must be greater than H, the force F may be found from the preceding equation by writing Z instead of L:

$$F = W\frac{\sin (Z + H)}{\sin (Z + K)} \qquad (1)$$

This value may be expressed in terms of the coefficient of friction f as follows:

$$F = W\frac{\sin(Z + H)}{\sin(Z + K)} = W\frac{\sin Z \cos H + \cos Z \sin H}{\sin Z \cos K + \cos Z \sin K}$$

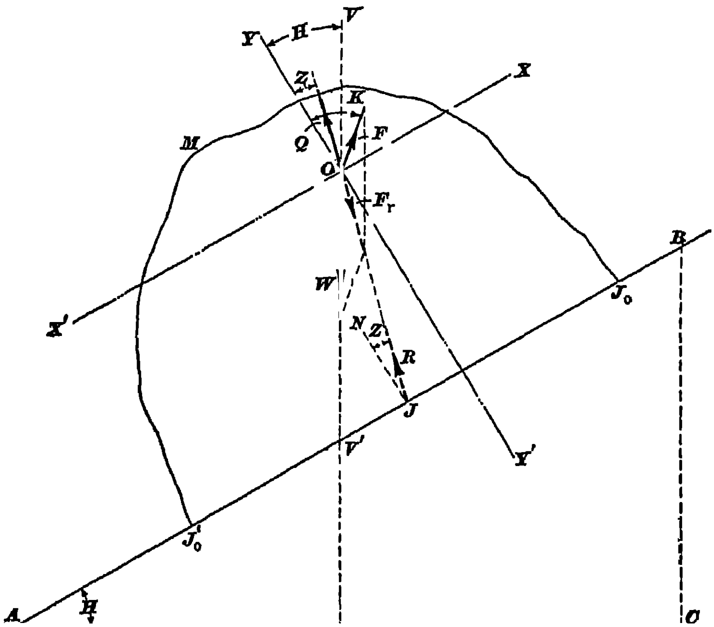


FIG 68

Dividing both terms of the fraction by cos Z, and writing f instead of $\frac{\sin Z}{\cos Z}$ (= tan Z, Art. **63**),

$$F = W\frac{f \cos H + \sin H}{f \cos K + \sin K} \qquad (2)$$

Leaving W, H, and f unchanged, it is seen from formula **1** that, if K is changed, F will have its least value

when the denominator is the greatest possible—that is, when $\sin(Z + K) = 1$; whence,

$$Z + K = 90°, \text{ and } 90° - K = Z, \text{ or } FOX = Z$$

This very important result is expressed by saying that *the best angle of traction up an inclined plane is the angle of friction.*

Case II.—*The body is on the point of moving down the plane* (Fig. 53).

In this case, the friction, which is the component parallel to the plane of the total reaction R, acts upwards. The line of action of R, and therefore of F_r, must lie on the right (in the figure) of the normal JN. By a process of reasoning similar to that employed in the preceding case, the following formulas are obtained for the present conditions:

$$F = W\frac{\sin(H - Z)}{\sin(K - Z)} \qquad (3)$$

$$F = W\frac{\sin H - f\cos H}{\sin K - f\cos K} \qquad (4)$$

75. Discussion of Formula 3 of Art. 74—Angle of Repose.—So long as H is greater than Z, formula **3** will give a positive value for F. In this case, F, being the equilibrant of Q and W, must act outside the angle WOQ, that is, K must be greater than Z. The positive value of F indicates that, if the body is left to itself, it will slide down the plane, and that, therefore, a force is necessary to keep it from sliding.

If $H = Z$, then $\sin(H - Z) = 0$, and, therefore, $F = 0$. This means that the body, if left to itself, will rest on the plane in a state of limiting equilibrium; no force is necessary to keep the body from sliding; but if the angle of the plane is increased and no force applied, the body will slide. For this reason, the angle of friction is often called the **angle of repose**, and defined as the greatest angle with the horizon that the surface of contact of the two bodies to whose friction it refers can make without the bodies sliding on each other This relation is made use of in the determination of the

coefficient of friction. Suppose the plane AB, Fig. 54, to be the upper surface of a plank hinged at A, and that it is desired to find the coefficient of friction between cast iron and marble. The plank is lined with a plate of cast iron and turned about the hinge until it is nearly horizontal. A block of marble is then laid on the plank, and the latter turned upwards until the block begins to slide. The angle of inclination of the

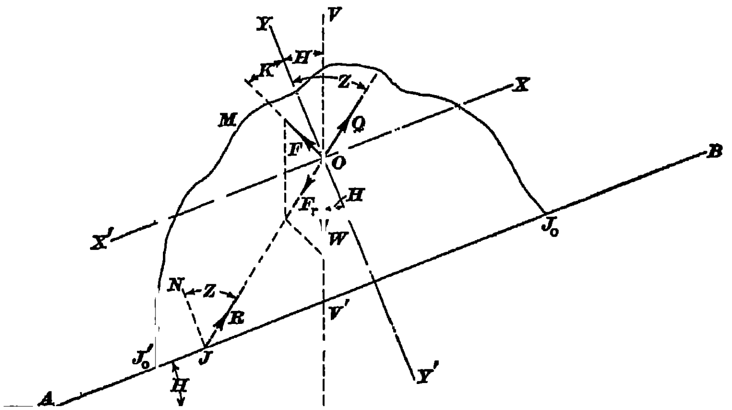


FIG. 54

plank to the horizontal at which this takes place is the angle of friction, and its tangent is the coefficient of friction.

When H is less than Z (see Fig 54), the component of W along the plane is not sufficient to overcome the friction. Therefore, in order that the body may be on the point of sliding down, the force F must have a component acting down the plane; that is, K must be on the left of YY' In this case,

$$F = W\frac{\sin(Z-H)}{\sin(Z+K)} \qquad (1)$$

or

$$F = W\frac{f\cos H - \sin H}{f\cos K + \sin K} \qquad (2)$$

EXAMPLE 1 —A wooden box 10 feet long, 6 feet wide, and 4 feet deep is used for carrying coal up and down an inclined steel-rail track, the grade of the track being 10 in 100 (which means that the track rises 10 feet for every 100 feet of length, measured horizontally). The weight of coal will be taken as 54 pounds per cubic foot, and the

coefficient of kinetic friction between wood and steel as .4. The box being full, required: (*a*) the magnitude and inclination of the least force that will keep the box moving up the plane with constant velocity; (*b*) the magnitude of a force parallel to the rails necessary to produce the same effect; (*c*) the magnitude of a force parallel to the rails that will keep the box moving downwards with constant velocity. (*d*) If the available force acting upwards, parallel to the track, is 4,000 pounds, to what depth can the box be filled?

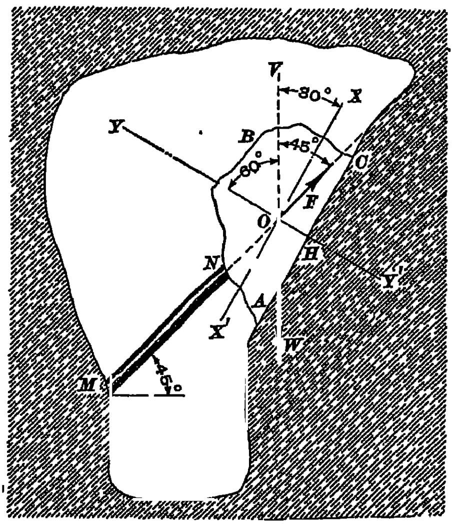


FIG. 55

SOLUTION.—(*a*) For the weight W we have, neglecting the weight of the box, $W = 10 \times 6 \times 4 \times 54 = 12{,}960$ lb. The angle Z of friction, to the nearest minute, is the angle whose tangent is .4, that is, $Z = 21° 48'$. For the inclination H of the track we have $\tan H = \frac{10}{100}$, whence $H = 5° 43'$. For the least force for which the box will be on the point of moving upwards, or that will keep the box, after the latter has been started, moving upwards with constant velocity, we must have (Art. **74**) $K = 90° - Z = 68° 12'$. These values in formula **1** of Art. **74** give

$$F = 12{,}960 \sin (5° 43' + 21° 48') = 5{,}988 \text{ lb. Ans.}$$

(*b*) In this case, $K = 90°$, and formula **1** of Art. **74** gives, noticing that $\sin (90° + Z) = \cos Z$,

$$F = 12{,}960 \frac{\sin (5° 43' + 21° 48')}{\cos 21° 48'} = 6{,}449 \text{ lb. Ans.}$$

(*c*) As here H is less than Z, formula **1** of Art. **75** is used, making (see Fig. 54) $K = YOX' = 90°$, and $\sin (Z + K) = \cos Z$.

$$F = 12{,}960 \frac{\sin (21° 48' - 5° 43')}{\cos 21° 48'} = 3{,}867 \text{ lb. Ans.}$$

(*d*) Let x be the height of the coal above the bottom of the box; then, $W = 10 \times 6 \times x \times 54$. Formula **1** of Art. **74** gives, noticing that here $F = 4{,}000$ and $K = 90°$,

$$W = F \frac{\sin (Z + K)}{\sin (H + Z)} = 4{,}000 \frac{\sin (21° 48' + 90°)}{\sin (5° 43' + 21° 48')} = 4{,}000 \frac{\cos 21° 48'}{\sin 27° 31'}$$

or, substituting the value of W just given,

$$10 \times 6 \times x \times 54 = 4{,}000 \frac{\cos 21^\circ 48'}{\sin 27^\circ 31'};$$

whence,

$$x = \frac{100}{81} \times \frac{\cos 21^\circ 48'}{\sin 27^\circ 31'} = 2\ 481 \text{ ft} = 2 \text{ ft } 5\tfrac{3}{4} \text{ in}, \text{ nearly.} \quad \text{Ans.}$$

EXAMPLE 2 —A piece of rock $A\,B\,C$, Fig 55, lying on the floor of a mine is kept from sliding down by a prop $M\,N$ Weight of $A\,B\,C$ is 3 tons; inclination of $C\,A$ to horizontal, 60°; inclination of $N\,M$, 45°; coefficient of friction, .75 Required the pressure on the prop.

SOLUTION —The pressure on the prop is equal and opposite to the reaction F of the prop The inclination of $N\,M$ to the horizontal being 45°, its inclination $V\,O\,F$ to the vertical is likewise 45°. Here, $H = 60^\circ$, therefore,

$K = Y\,O\,F = 60^\circ + 45^\circ = 105^\circ$, $\sin K = \cos 15^\circ$, $\cos K = -\sin 15^\circ$

Formula 4 of Art 74 gives

$$F = 3 \times \frac{\sin 60^\circ - \frac{3}{4} \times \cos 60^\circ}{\cos 15^\circ + \frac{3}{4} \sin 15^\circ} = 3 \times \frac{.866 - \frac{3}{4} \times .5}{.966 + \frac{3}{4} \times .259} = 1.27 \text{ T.} \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE

1. What force will keep an iron block weighing 4 tons, placed on an iron plate inclined at 45° to the horizontal, in a condition of limiting equilibrium with respect to upward motion: (*a*) if the force is parallel to the plate? (*b*) if the force makes with the plate an angle of 30°? Take $f = .20$.

NOTE —In this case, it is more convenient to use formulas involving f, rather than Z.

$$\text{Ans.} \begin{cases} (a) & F = 3.394 \text{ T.} \\ (b) & F = 3.513 \text{ T.} \end{cases}$$

2 If, in the preceding example, the force is parallel to the plate and acts upwards, what must its magnitude be, that the block may be on the point of moving downwards? Ans $F = 2.263$ T.

3. Taking the coefficient of kinetic friction between steel and pine as .16, what is the least force that can keep a steel block having a weight of 2 tons moving with constant velocity up a pine plank inclined at 20° to the horizontal? Ans $F = 1{,}944$ lb

4. A block of marble weighing 1,000 pounds is to be kept moving with constant velocity up an inclined white-pine plank by a force of 600 pounds; what must be the inclination of the plank, assuming that the best angle of traction is used, and that $f = .45$?

Ans. $H = 12^\circ 38'$

KINEMATICS AND KINETICS

COMPOSITION AND RESOLUTION OF VELOCITIES

1. Graphic Representation of Velocity.—Velocity, like force, is a *vector quantity*—that is, a quantity having both magnitude and direction—and, like all vector quantities, can be represented by a straight line, called a *vector* (see *Fundamental Principles of Mechanics*). The vector is drawn parallel to the direction of the velocity represented; its length is made, to any convenient scale, equal to the magnitude of that velocity, and the arrowhead on the vector is placed so that it will point in the direction of the motion under consideration.

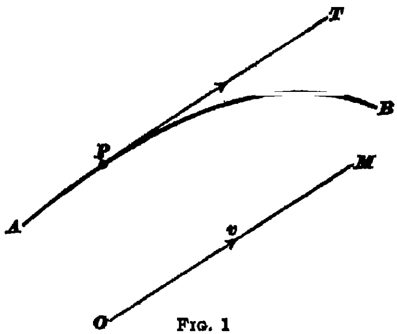


FIG. 1

In Fig 1, let AB be the path of a moving point, and v the velocity the point has when it occupies the position P on its path. The direction of the motion at the instant considered is that of the tangent PT to the curve AB. The velocity v may be represented by the vector OM drawn parallel to PT through any convenient point. If v is expressed in feet per second, and a scale of 5 feet per second to the inch is adopted, the length of the vector OM should be $\frac{v}{5}$. Similarly for any other scale.

2. Parallelogram of Velocities.—Let a body or particle O, Fig. 2, be moving in the direction $O X_2$ on a flat surface $A B$, with a velocity v_2 relative to that surface, that is, in such a manner that, if the line $O X_2$ is fixed on the surface $A B$, the particle O will move in that line describing v_2 units of length per unit of time. Let P_2 be the position of the moving particle after the time t. Then,

$$O P_2 = v_2 t \qquad (1)$$

While the particle has this motion, let the surface $A B$ move in the direction $O X_1$ with uniform velocity v_1, and let $A' B'$ be

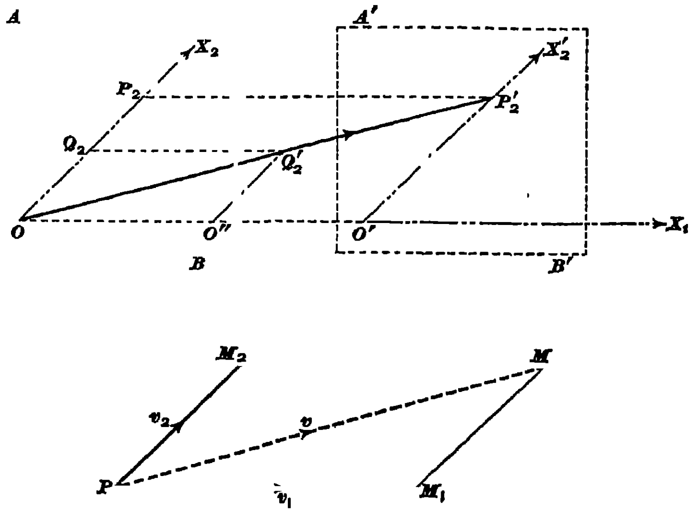


FIG 2

the position of the surface after the time t, O' being the corresponding position of O, and $O' X_2'$, parallel to $O X_2$, the corresponding position of $O X_2$. Then,

$$O O' = v_1 t \qquad (2)$$

The moving particle will now be at P_2', the distance $O' P_2'$ being equal to $O P_2$, or $v_2 t$. The figure $O O' P_2' P_2$ is evidently a parallelogram.

In the same manner, it can be shown that, after a time t', in which the moving particle has described the distance $O Q_2$ along the line $O X_2$, while the surface has moved so

that O'' is the position of the starting point O, the final position Q_2' of the particle will be the end of the diagonal of the parallelogram $O\,O''\,Q_2'\,Q_2$, in which

$$O''\,Q_2' = O\,Q_2 = v_2\,t' \qquad (3)$$

$$O\,O'' = Q_2\,Q_2' = v_1\,t' \qquad (4)$$

Dividing (2) by (1) gives

$$\frac{v_1}{v_2} = \frac{O\,O'}{O\,P_2} = \frac{O\,O'}{O'\,P_2'}$$

and dividing (4) by (3) gives

$$\frac{v_1}{v_2} = \frac{O\,O''}{O''\,Q_2'}$$

Equating these two values of $\frac{v_1}{v_2}$,

$$\frac{O\,O'}{O'\,P_2'} = \frac{O\,O''}{O''\,Q_2'}$$

According to the theory of similar triangles, the last equation shows that O, Q_2', and P_2' are in the same straight line. It follows, therefore, that, at every instant, the moving particle is on the straight line $O\,P_2'$—or, what is the same thing, that the particle moves in that line. The spaces $O\,P_2'$, $O\,Q_2'$ described by the particle in times t and t', respectively, are to each other as $O\,O'$ is to $O\,O''$—that is, as $v_1\,t$ is to $v_1\,t'$, or as t is to t'. Therefore, the motion of the particle along $O\,P_2'$ is a uniform motion. If $O\,O''$ represents v_1, and $O\,Q_2$ represents v_2, the diagonal $O\,Q_2'$ will evidently represent the space described by the particle in a unit of time—that is, the velocity v of the particle along its path $O\,P_2'$. Hence, the following construction:

From any point P, draw two vectors PM_1 and PM_2, representing, to any convenient scale, the velocities v_1 and v_2, respectively. Construct a parallelogram $PM_1\,MM_2$ on those two vectors. Then will the diagonal PM represent, to the scale adopted, the velocity of the moving particle in its path.

3. It will be observed that the velocity v is determined by the same general method used for finding the resultant of two concurrent forces. With respect to the velocities v_1 and v_2, the velocity v, which is the actual velocity of the

particle, is called the **resultant velocity**, and v_1 and v_2 are called the **components** of v in the directions OX_1 and OX_2, respectively. Since the particle has the two velocities v_1 and v_2 at the same time, these velocities are said to be **simultaneous.**

The principle of the **parallelogram of velocities**, which is similar to that of the parallelogram of forces, may be stated as follows:

If two simultaneous velocities of a particle are represented in magnitude and direction by two vectors drawn from the same origin, and a parallelogram is constructed on these two vectors, the resultant velocity is represented in magnitude and direction by that diagonal of the parallelogram that passes through the common origin of the two vectors, this diagonal being treated as a vector having the same origin

4. The process of finding the resultant of two or more simultaneous velocities is called **composition of velocities.**

As in the case of forces, any velocity may be considered to be the resultant of two velocities in any given directions. Thus, the velocity v, Fig. 3, may be considered as the resultant of the simultaneous velocities v_1 and v_2 in the direction OX_1 and OX_2, respectively; or as the resultant of the simultaneous velocities v_1' and v_2' in the direction OX_1' and OX_2', respectively, etc. To **resolve** a velocity into its components in given directions is to find the values of these components, or to replace the given velocity with these components, the process whereby this is accomplished is called **resolution of velocities.**

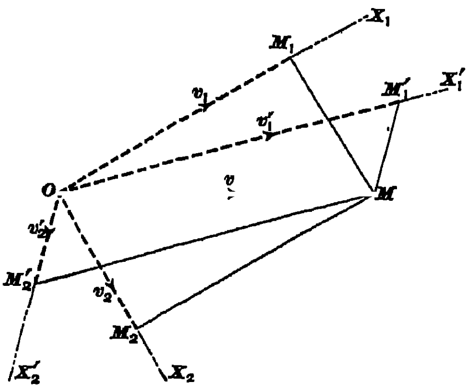


FIG 3

5. Since velocities are combined and resolved in the same manner as forces, all that has been said relating to the composition and resolution of forces applies to the composition and resolution of velocities. Thus, instead of the parallelogram of velocities, the **triangle of velocities** is often used. In Fig. 2, for example, the resultant v may be determined by drawing PM_1 to represent v_1, and then M_1M to represent v_2, and drawing PM. The magnitude of PM can be ascertained either graphically (by constructing the parallelogram or the triangle accurately to scale), or analytically (by applying the principles of trigonometry), as in the case of forces.

EXAMPLE 1 —A ship is propelled by its screw in a northeast direction at the rate of 15 knots, while the current carries it due south at the rate of 5 knots Find the resultant motion of the ship (A knot is a velocity of 1 nautical mile, or 6,080 feet, per hour)

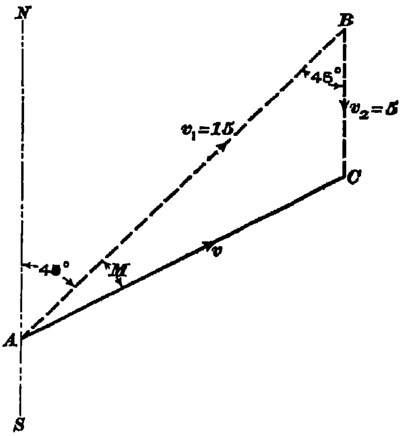


FIG 4

GRAPHIC SOLUTION —Draw the north-and-south line NS, Fig 4 From any point A and to any convenient scale, draw $AB = 15$, making an angle of 45° with AN. This will represent the velocity of 15 knots toward the northeast From B draw BC due south, that is, parallel to NS, and equal to 5, using the same scale as before Join AC, and mark the arrowhead so that it will be in non-cyclic order with AB and BC. The length of AC, measured to the scale used for v_1 and v_2, will give the magnitude of the resultant velocity. The direction is determined by measuring the angle NAC with a protractor. Ans.

ANALYTIC SOLUTION —The triangle ABC gives

$$v = AC = \sqrt{15^2 + 5^2 - 2 \times 5 \times 15 \cos 45^\circ} = 12 \text{ knots}$$

Also, $$\sin M = \frac{v_2 \sin 45^\circ}{v} = \frac{5 \sin 45^\circ}{12}$$

The angle M, taken to the nearest minute, may be either 17° 8′ or 180° − 17° 8′ = 162° 52′ As BC, opposite M, is the shortest side of the triangle, the value 17° 8′ must be taken. Then, NAC

$= 45° + 17° 8' = 62° 8'$ The ship, therefore, is moving in a direction N 62° 8′ E, at the rate of 12 knots, nearly Ans.

EXAMPLE 2.—A river 4 miles wide has a current velocity of 3 miles per hour A boat whose paddle wheels can carry it through 6 miles per hour in still water is to cross the river from a point A, Fig. 5, so as to land at a point B directly opposite the starting point A. It is required to find: (a) the direction in which the boat must be headed, (b) the time required for a trip across the river

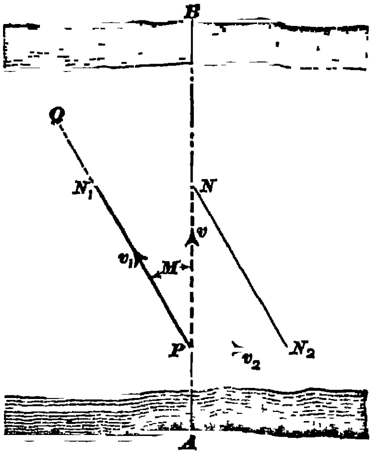


FIG 5

SOLUTION —(a) Since the velocity of the current and that imparted to the boat by the paddle wheels are both uniform, the boat must move along the line AB with a uniform velocity v (to be determined). Let P be the position of the boat at any instant, and PQ the direction in which it is headed, and let v_1 be the velocity imparted to it in that direction by the paddle wheel. In addition to v_1, the boat has a velocity v_2 down the stream, equal to the velocity of the current. The velocity v is the resultant of the velocities v_1 and v_2. The triangle PNN_1, being right-angled at N, gives

$$v = PN = \sqrt{\overline{PN_1}^2 - \overline{N_1N}^2} = \sqrt{v_1^2 - v_2^2} = \sqrt{6^2 - 3^2}$$
$$= 5.196 \text{ mi. per hr.}$$

Also, $\sin M = \dfrac{N_1N}{PN_1} = \dfrac{v_2}{v_1} = \dfrac{3}{6} = \dfrac{1}{2}$; $M = 30°$. Ans.

(b) The time required is

$$\frac{AB}{v} = \frac{4}{5.196} = .77 \text{ hr.} = 46 \text{ min., nearly.} \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE

1. A point has two simultaneous velocities, one of 100 feet per second and one of 200 feet per second. The vectors representing these two velocities make an angle of 45° with each other. Find: (a) the resultant velocity v; (b) its inclination M to the velocity of 200 feet. (Angles are given to the nearest 10 seconds.)

Ans. $\begin{cases} (a)\ v = 279.79 \text{ ft. per sec.} \\ (b)\ M = 14° 38' 20'' \end{cases}$

2. A balloon moves upwards with a velocity of 50 feet per second, and at the same time the wind carries it in a horizontal direction at the

rate of 20 feet per second Find (*a*) the resultant velocity v, (*b*) its inclination M to the vertical

Ans $\begin{cases} (a)\ v = 53\ 852 \text{ ft per sec.} \\ (b)\ M = 21° 48' 10'' \end{cases}$

6. Absolute and Relative Velocity.—Velocity, like motion, is always relative, it represents the rate of motion of a body with respect to another, and the same velocity can have different values according to the condition of the objects to which it is referred. Thus, when a locomotive is running, the velocity of the piston with respect to the cylinder in which it moves is the space that the piston describes in the cylinder per unit of time, the velocity of the piston with respect to the ground is equal to the velocity of the piston with respect to the cylinder added to or subtracted from the velocity that, in common with the whole engine, the cylinder has with reference to the ground—added, if piston and engine are moving in the same direction; otherwise, subtracted.

7. In nearly all practical questions, it is customary to refer velocities to the surface of the earth. When the velocity of a body is given without any qualification, it is usually understood to be the velocity of the body relative to the ground. This velocity is customarily, although not properly, called **absolute velocity**; while the term **relative velocity** is restricted to indicate velocity with respect to objects that are themselves in motion with respect to the ground. Thus, in the example of the preceding article, the relative velocity of the piston is its velocity with respect to the cylinder; while the absolute velocity of the piston is its velocity with respect to the ground. In the case represented in Fig. 2, the velocity of the moving particle, relative to the surface AB, is v_2; the absolute velocity of the particle is v; the absolute velocity of AB is v_1. It is obvious that, if a body moves on another with a certain relative velocity, while the latter is in motion, the absolute velocity of the former body is the resultant of its relative velocity and the absolute velocity of the other body.

UNIFORM MOTION IN A CIRCLE

ANGULAR VELOCITY

8. Angular Displacement.—Let a point P, Fig. 6, be moving uniformly in a circular path of radius r. The center of the circle is C, and the velocity of the moving point is v. If the point moves from a position P to a position P_1 in the time t, the angle H swept over by the radius in passing from the position CP to the position CP_1 is called the **angular displacement** of P with respect to C. Angular displacement may be measured in degrees, but is usually measured in radians. As explained in *Geometry*, Part 2, the radian measure of the angle H is $\frac{\text{arc } PP_1}{r}$, or

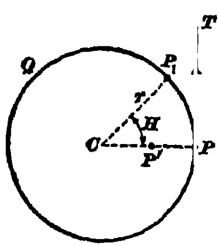


FIG. 6

$$H \text{ (in radians)} = \frac{\text{arc } PP_1}{r}$$

If the length of the arc PP_1 is denoted by s,

$$H = \frac{\text{arc } PP_1}{r} = \frac{s}{r} \qquad (1)$$

Also,
$$s = rH \qquad (2)$$

9. Angular Velocity.—Since the velocity v is uniform, $s = vt$; that is, $rH = vt$; whence

$$H = \frac{v}{r}t \qquad (1)$$

The angular displacement is, then, proportional to the time, and $\frac{v}{r}$ is evidently the angular displacement per unit of time, since, when t is made equal to 1 in the preceding formula, H becomes equal to $\frac{v}{r}$. This displacement per unit

of time is called the **angular velocity** of the moving point. The angular velocity may be defined also as the angle described by the moving point in a unit of time. As already stated, it is customary to express angular velocity in radians per unit of time. It is also customary to represent this quantity by the Greek letter ω (*o-me′-ga*). Therefore (see formula **1**),

$$\omega = \frac{H}{t} \qquad (2)$$

10. Relations Between Angular and Linear Velocity.—The velocity of a moving point in the direction of the tangent to its path—that is, what in previous articles has been called *the* velocity of the point—is often called **linear** or **tangential velocity**, in order to distinguish it from angular velocity. When the word velocity is used without any qualification, linear velocity is meant The velocity in the direction of the path, especially when the latter is a circle, is also called **circumferential velocity.**

From Art. **9**, the following obvious and very important relation between linear and angular velocity is obtained:

$$\omega = \frac{v}{r} \qquad (1)$$

$$v = r\,\omega \qquad (2)$$

If two points move in two circles of radii r_1 and r_2 with the same angular velocity ω, and linear velocities v_1 and v_2, then, $v_1 = r_1\,\omega$, and $v_2 = r_2\,\omega$; whence, by division,

$$\frac{v_1}{v_2} = \frac{r_1}{r_2} \qquad (3)$$

that is, *the linear velocities of two points moving in different circles with the same angular velocity are directly proportional to the radii of the respective paths.*

11. If the linear velocities are the same, and the angular velocities are ω_1 and ω_2, we must have $v = r_1\,\omega_1 = r_2\,\omega_2$; whence,

$$\frac{\omega_1}{\omega_2} = \frac{r_2}{r_1}$$

that is, *for the same linear velocity, the angular velocities of two moving points are inversely as the corresponding radii of the paths.*

12. If, in Fig. 6, the radius CP is imagined to move with P, all the points in CP will evidently have the same angular velocity, since they will all describe the same angle in the same time. If the linear velocity of the point P', situated at unit's distance from the center, is v', then,

$$v' = 1 \times \omega = \omega$$

The angular velocity of P may, therefore, be defined as being the linear velocity of, or the length of the arc described in a unit of time by, a point on the radius CP situated at unit's distance from the center, when the radius is considered as moving with the point P.

13. Angular Velocity in Terms of Number of Revolutions Per Unit of Time.—In practical engineering problems, it is customary to state the velocity of circular motion in terms of the number of revolutions per unit of time (usually per minute); that is, the number of times that the moving point passes over an entire circumference in a unit of time. If this number is denoted by n, and the radius of the circle by r, the space passed over by the point in a unit of time is $2\pi r \times n$; hence, for the linear velocity of the point, we have

$$v = 2\pi r n$$

and for its angular velocity (formula **2** of Art. **10**),

$$\omega = \frac{v}{r} = 2\pi n$$

If n is the number of revolutions per minute, the velocities v and ω, referred to the second, are:

$$v = \frac{2\pi r n}{60} = \frac{\pi r n}{30} = .10472\,r n \qquad (1)$$

$$\omega = \frac{2\pi n}{60} = \frac{\pi n}{30} = .10472\,n \qquad (2)$$

Conversely, if v or ω, referred to the second, is given, the number of revolutions per minute is given by the formula

$$n = \frac{30\,v}{\pi r} = 9.5493\,\frac{v}{r} = 9.5493\,\omega \qquad (3)$$

14. Angular Velocity of a Rotating Body.—Whenever any body, as a wheel, revolves about a fixed axis, every

point in the body describes a circle whose radius is equal to the distance of the point from the axis. Since all points revolve through the same angle in the same time, they all have the same angular velocity. This common angular velocity is called the angular velocity of the rotating body.

EXAMPLE 1 —A drum W, Fig. 7, whose radius R is 1.5 feet is fixed to a revolving shaft S. The motion of W is transmitted to another shaft s by means of a belt $B\ B$ passing around the drum W and a drum w fixed on the shaft s. If the velocity of the belt is 10 feet per second and the drum w makes 100 revolutions per minute, it is required to determine the number N of revolutions that S or W makes per minute, the radius r of the drum w, and the angular velocity of the two drums in radians per second. It is assumed that the circumferential velocity of each drum is the same as the velocity of the belt

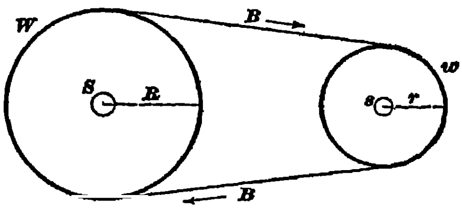


FIG 7

SOLUTION —The circumferential velocities of the two drums, in feet per second, are, respectively, $\frac{\pi R N}{30}$ and $\frac{\pi r n}{30}$ (Art 13). Since each of these velocities is equal to the velocity of the belt,

$$\frac{\pi R N}{30} = 10 \qquad (1)$$

$$\frac{\pi r \times 100}{30} = 10 \qquad (2)$$

From (1) we get (see also formula 3 of Art. 13), replacing R by its value 1.5,

$$N = 9.5493 \times \frac{10}{1.5} = 63\ 662 \quad \text{Ans.}$$

And from (2) (see the same formula),

$$r = 9.5493 \times \frac{10}{100} = .95493 \text{ ft.} \quad \text{Ans.}$$

For the angular velocity ω_1 (radians per second) of W, formula 1 of Art. 10 gives

$$\omega_1 = \frac{10}{R} = \frac{10}{1.5} = 6\ 6667. \quad \text{Ans.}$$

And for the angular velocity ω_2 of w (formula 2 of Art. 13),

$$\omega_2 = \frac{100\,\pi}{30} = 10.472. \quad \text{Ans.}$$

EXAMPLE 2 —Two drums D and D', Fig 8, of radii r and r' feet, are fixed on a shaft $S\,S$ A rope $M\,N\,L\,Q$ is wound around the drums in opposite directions, so that when $M\,N$ rises and winds around D, the

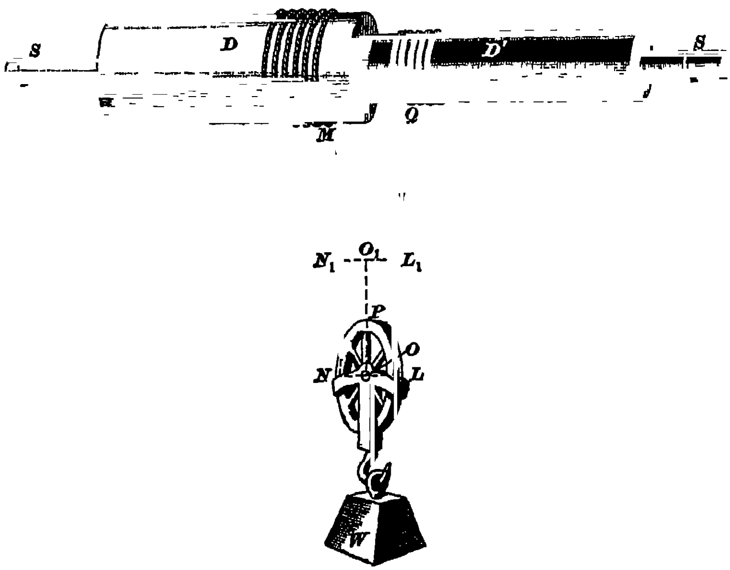


FIG 8

portion $Q\,L$ descends by unwinding from D'. The rope passes around a pulley P carrying a weight W. It is required to find the motion of the weight W when the shaft $S\,S$ revolves at the rate of n revolutions per minute

NOTE —This arrangement is known by the various names of **differential windlass, differential wheel and axle,** and **Chinese wheel and axle,** and is used to obtain a slow motion of W, which, under some conditions, is a mechanical advantage

SOLUTION.—It is assumed that every turn of the rope around the drums is a circle, and that $M\,N$ and $Q\,L$ remain parallel. This is near enough to the actual conditions occurring in practice

Let the shaft turn in the direction indicated by the curved arrows. Then $N\,M$ will rise and $Q\,L$ will descend For the angular velocity of the shaft and the drums (formula **2** of Art **13**), we have $\omega = 10472\,n$ radians per sec. Let v and v' be the velocities of $N\,M$ and $Q\,L$, or of M and Q, respectively, in feet per second. Then (formula **2** of Art. **10**),

$$v = r\,\omega = 10472\,n\,r$$
$$v' = r'\,\omega = .10472\,n\,r'$$

During any time t, the center O of the pulley, and therefore the weight W, moves through a distance $O O_1$, which is easily determined. The length of the rope between M and Q is reduced by an amount equal to $N N_1 + L L_1 = 2\, O O_1$ But, as during this time a portion $v\,t$ of the rope has wound about D, and a portion $v'\,t$ unwound from D',

$$2\, O O_1 = v\,t - v'\,t,$$

whence

$$O O_1 = \tfrac{1}{2}(v - v')t$$

Therefore the point O and the load W move uniformly with a velocity v_w equal to $\frac{1}{2}(v - v')$ Substituting the values of v and v' found above,

$$\begin{aligned} v_w &= \tfrac{1}{2}(r - r')\omega = \tfrac{1}{2}(\ 10472\, n\, r - .10472\, n\, r') \\ &= 05236\, n(r - r') \quad \text{Ans} \end{aligned}$$

EXAMPLES FOR PRACTICE

1. A flywheel 6 feet in diameter makes 75 revolutions per minute Find· (*a*) its angular velocity ω, in radians per second, (*b*) the velocity v of a point on the rim, in feet per second

$$\text{Ans}\begin{cases}(a)\ \omega = 7\ 854 \\ (b)\ v = 23.562\end{cases}$$

2. A point revolves in a circle 8 feet in diameter with an angular velocity of 4 5 radians per second Find: (*a*) its linear velocity v, (*b*) the number n of revolutions it makes per minute.

$$\text{Ans.}\begin{cases}(a)\ v = 18 \text{ ft. per sec.} \\ (b)\ n = 42\ 972\end{cases}$$

3. Two drums A and B on parallel shafts are driven by a belt traveling at the rate of 1,000 feet per minute; the radius of A is 5 feet, and B makes 300 revolutions per minute. Find. (*a*) the number n of revolutions that A makes per minute; (*b*) the angular velocities ω_a and ω_b of the two drums, in radians per second, (*c*) the radius r of B

$$\text{Ans.}\begin{cases}(a)\ n = 31.831 \\ (b)\begin{cases}\omega_a = 3\ 333 \\ \omega_b = 31.416\end{cases} \\ (c)\ r = .531 \text{ ft.}\end{cases}$$

CENTRIPETAL AND CENTRIFUGAL FORCE

15. Restatement of the Law of Inertia.—A provisional statement of the law of inertia was made in *Fundamental Principles of Mechanics*. Now that the theory of the center of gravity has been explained, the law can be expressed in a more complete and definite manner as follows:

If a body is under the action of no force, its center of gravity is either at rest or moving uniformly in a straight line.

16. Centripetal Force.—Let a body be moving in such a manner that its center of gravity G, Fig. 9, travels uniformly in a circle $A B$ of radius r. As examples of this kind of motion may be mentioned a car moving in a curved track, and the balls of a ball governor. According to the law of inertia, such motion cannot take place unless the body is acted on by unbalanced forces, for, were the body under the action of no force, the path of its center of gravity would be a straight line instead of a circle It can be shown by the use of advanced mathematics that, in order to preserve this motion, the body must be constantly acted on by a force P whose line of action is directed toward the center of the circle and passes through the center of gravity of the body. This force, which is exerted *on* the revolving body, is called **centripetal force.**

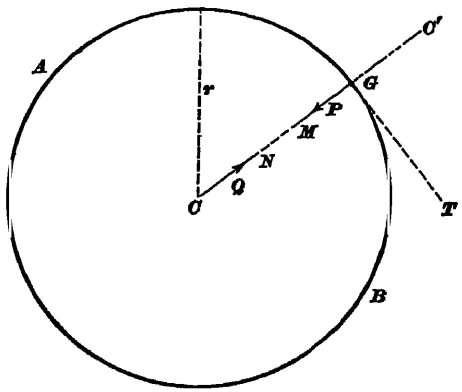


FIG 9

If the mass of the body is denoted by m, and the linear velocity of its center of gravity by v, the magnitude of the centripetal force P is given by the formula

$$P = \frac{m v^2}{r}$$

Since $m = \frac{W}{g}$, where W denotes the weight of the body, and g the acceleration of gravity, we have also,

$$P = \frac{W v^2}{g r} \qquad (1)$$

In applying this formula, v should be expressed in feet per second, and r in feet, since g is referred to the foot and second as units.

If, instead of v, the angular velocity ω is given, we have, from Art. **10**, $v = r\omega$, $v^2 = r^2\omega^2$. Substituting this value in formula **1**, and reducing,

$$P = \frac{W r \omega^2}{g} \qquad (2)$$

To express P in terms of the number n of revolutions per minute, we have (Art. **13**), $\omega = \frac{\pi n}{30}$, and, therefore, substituting in formula **2**,

$$P = \frac{W r \pi^2 n^2}{900\, g} \qquad (3)$$

If the revolving body is acted on by any system of forces, their resultant must pass through the center of gravity of the body, be directed toward the center of the circle, and have the magnitude given by any of the preceding formulas.

17. Illustrations of Centripetal Force.—A familiar example of centripetal force is afforded by the circular motion of a ball tied to a string and swung around, the other end of the string being held in the hand. The ball constantly tends to move along the tangent, or "fly off on the tangent," and would do so, if it were not kept in the circular path by the pull of the string.

It should be noticed that, while the centripetal force is always directed toward the center of the circle, it is not necessarily exerted from that center; in other words, the body to the action of which the centripetal force is due does not need to be at the center. In the case of the ball given above, the hand, which exerts the centripetal force, is placed at the center of the circle. In the case of a train moving on a curved track, on the contrary, the centripetal force is the resultant of the weight of the car and the pressure of the rails on the wheels; and this force, although directed toward the center of the curve, is exerted at the circumference.

18. Centrifugal Force.—By the law of action and reaction, a particle (or body) moving in a circle must exert on the body to whose action the centripetal force is due a reaction equal and opposite to that force. This reaction is

called **centrifugal force.** Thus, in the case of the ball considered in the preceding article, the pull exerted on the hand through the string is the centrifugal force exerted by the ball on the hand.

Suppose the source of the centripetal force P, Fig. 9, to be at the center C of the circle. The force P that acts on the body G may be represented by the vector GM, while the centrifugal force Q, which is exerted by the body G, may be represented by the equal and opposite vector CN.

19. Caution Against Some Common Misconceptions Regarding Centrifugal Force.—The following facts must be clearly understood, as this subject is one about which students (and even teachers) often fall into very gross errors:

In studying the motion of a body in a circular path, the centripetal, not the centrifugal, force must be considered as the only force acting on the body. In Fig. 9, the body G is under the action of the force P, not under the action of the force Q.

A body moving in a circular path has no tendency to move along the radius, its tendency being to move along the tangent. In Fig. 9, the body G, if left to itself, would move along GT, not along GC'.

A body moving in a circle is at every instant under the action of an unbalanced force (the centripetal force); as, otherwise, the center of gravity of the body would move in a straight line.

Although the centripetal and the centrifugal force are collinear, equal in magnitude and opposite in direction, they do not act on the same body, and cannot, therefore, balance each other. It should be kept in mind that, when it is said that two or more forces balance, the meaning of the statement is that they produce no motion on the body on which they all act; and, if several forces act on different independent bodies, it cannot be said that they form either a balanced or an unbalanced system, as, under such circumstances, the

expressions balanced system and unbalanced system have absolutely no meaning.

EXAMPLE —A ball B, Fig 10, whose weight is W, is suspended by a string of length l from a point O, and made to revolve uniformly in a horizontal circle MN, at the rate of n revolutions per minute It is required to determine (a) the radius r of the circle MN, (b) the centripetal force P acting on the ball, (c) the tension T in the string

NOTE —This contrivance is called a **conical pendulum**, and its operation is very similar to that of the ball governor of an engine

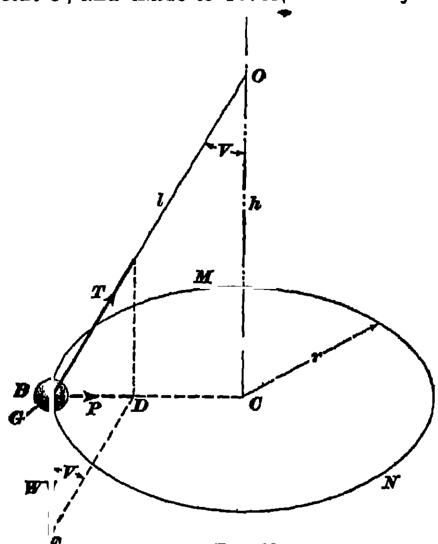


FIG 10

SOLUTION —(a) The two forces acting on the ball are the weight W of the ball and the tension T of the string. According to the principles stated in the preceding articles, the resultant P of these two forces must be directed toward the center C of the circle, and have the magnitude given by formula **3** of Art **16**; that is,

$$P = \frac{W r \pi^2 n^2}{900 g} \qquad (1)$$

In the triangle $B E D$,

$$P = W \tan V,$$

or, since $\tan V = \tan B O C = \frac{r}{h} = \frac{r}{\sqrt{l^2 - r^2}}$,

$$P = \frac{W r}{\sqrt{l^2 - r^2}} \qquad (2)$$

Equating the second members of (1) and (2),

$$\frac{W r \pi^2 n^2}{900 g} = \frac{W r}{\sqrt{l^2 - r^2}};$$

whence

$$\sqrt{l^2 - r^2} = \frac{900 g}{\pi^2 n^2}$$

Squaring and solving for r,

$$r = \sqrt{l^2 - \left(\frac{900 g}{\pi^2 n^2}\right)^2}. \quad \text{Ans.}$$

(*b*) Substituting in equation (2) the values just found for $\sqrt{l^2 - r^2}$ and r,

$$P = \frac{W\sqrt{l^2 - \left(\frac{900\,g}{\pi^2 n^2}\right)^2}}{\frac{900\,g}{\pi^2 n^2}} = \frac{W\sqrt{(\pi^2 n^2 l)^2 - (900\,g)^2}}{900\,g}$$

$$= W\sqrt{\left(\frac{\pi^2 n^2 l}{900\,g}\right)^2 - 1} \quad \text{Ans}$$

(*c*) In the triangle BDE,

$$T = ED = \sqrt{W^2 + P^2},$$

or, since $P^2 = W^2\left[\left(\frac{\pi^2 n^2 l}{900\,g}\right)^2 - 1\right]$,

$$T = \sqrt{W^2 + W^2\left[\left(\frac{\pi^2 n^2 l}{900\,g}\right)^2 - 1\right]} = W \times \frac{\pi^2 n^2 l}{900\,g}. \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE

1 A ball weighing 50 pounds revolves uniformly on a smooth horizontal surface, about an axis to which the ball is connected by a steel rod (*a*) If the rod is 10 feet long and the ball makes 90 revolutions per minute, what is the tension in the rod? (*b*) If the greatest tension that the rod can stand is 2,500 pounds, what is the greatest number of revolutions per minute that the ball can make without breaking the rod? Ans $\begin{cases} (a) & 1{,}381 \text{ lb.} \\ (b) & 121 \end{cases}$

2 The height h ($= CO$, Fig 10) of a conical pendulum is 2 feet. How many revolutions per minute can the pendulum make? (Observe that the number of revolutions is independent of the weight of the pendulum and of the length of the suspending string.) Ans. 38.3

3 If, in example 2, the weight of the pendulum is 10 pounds and the length of the string 16 feet, find (*a*) the centripetal force P; (*b*) the tension T in the string. Ans. $\begin{cases} (a) & 70.4 \text{ lb.} \\ (b) & 80 \text{ lb.} \end{cases}$

MOTION OF A TRAIN ON A CURVED TRACK

20. General Theory.—In Fig. 11 are represented the rails R_1 and R_2 of a curved track, and the flanges F_1 and F_2 of two directly opposite wheels of a car. The inclination H, or $R_1 R_2 N$, of the track to the horizontal is shown very much exaggerated for the sake of clearness. The weight W that comes on the two wheels acts through G, at a height a above

the track equal to the height of the center of gravity of the car. The difference NR_1 in elevation between the outer rail R_1 and the inner rail R_2 is called the **superelevation** of the outer rail When the car is moving uniformly in the curve, G moves in a horizontal circle whose radius is practically the same as the radius of the center line of the track. According to the theory of centripetal force, the resultant P of all the forces acting on that part of the car whose weight is carried by the two wheels here considered must pass through G

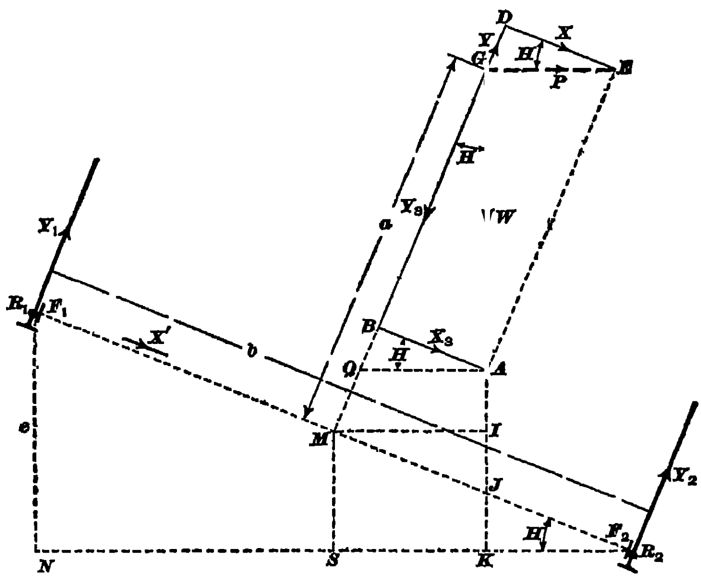


FIG 11

and be directed toward the center of the circle in which G moves; that is, P must be horizontal. For convenience all forces will be resolved into components parallel and perpendicular to $R_1 R_2$. The components of P in these directions are denoted by X and Y, as shown in the triangle GDE.

The forces acting on the car (or on the part here considered) are W, which is resolved into the component X_3 and Y_3, and the pressures of the rails. The components of these

pressures perpendicular to $R_1 R_2$ are denoted by Y_1 and Y_2; the component parallel to or along $R_1 R_2$ is denoted by X'. If, as shown in the figure, X' acts toward R_2, the pressure is exerted by the rail R_1 on the flange F_1; if X' acts toward R_1, the pressure is exerted by R_2 on the flange F_2. In the general derivation of the formulas, X' will be treated as acting toward R_2; if, in any particular case, the value of X' is found to be negative, this means that X' acts in the opposite direction, and that it is, therefore, exerted by the lower rail R_2.

Let v = linear velocity of car, in feet per second,
e = superelevation of outer rail, in feet;
$b = R_1 R_2$ = distance between centers of rails, in feet;
r = radius of center line of track, in feet.

According to the principles of statics, we have, since P is the resultant of the forces acting on the car,

$$X' + X_2 = X \qquad (1)$$

$$Y_1 + Y_2 - Y_3 = Y \qquad (2)$$

From (1), $$X' = X - X_2 \qquad (3)$$

Now,

$$X = P \cos H = \frac{Wv^2}{gr} \cos H \text{ (Art. 16)} = \frac{Wv^2}{gr} \times \frac{NR_2}{b}$$

$$= \frac{Wv^2 \sqrt{b^2 - e^2}}{grb}$$

$$X_2 = W \sin H = W \frac{NR_1}{R_1 R_2} = \frac{We}{b}$$

These values in (3) give

$$X' = \frac{W}{b}\left(\frac{v^2}{gr}\sqrt{b^2 - e^2} - e\right) \qquad (1)$$

To find Y_1, moments are taken about R_2. Since the moments of X' and Y_2 about R_2 are each zero, the algebraic sum of the moments of Y_1 and W must be equal to the moment of P; that is,

$$Y_1 b - W \times KR_2 = P \times GK \qquad (4)$$

The figure gives, MI being horizontal and MS vertical,

$$KR_2 = JR_2 \cos H = (MR_2 - MJ) \cos H$$

$$= (\tfrac{1}{2} b - MG \tan H) \cos H = \tfrac{1}{2} b \cos H - a \sin H$$

$$GK = GI + IK = GI + MS = a \cos H + \tfrac{1}{2} b \sin H$$

Substituting these values in (4), writing $\frac{Wv^2}{gr}$ for P, and solving for Y_1,

$$Y_1 = \frac{W}{b}\left[\left(\tfrac{1}{2}b + a\frac{v^2}{gr}\right)\cos H + \left(\tfrac{1}{2}b\frac{v^2}{gr} - a\right)\sin H\right] \qquad (2)$$

The value of Y_2 is found from equation (2) by substituting the expressions for the values of Y, Y_1, and Y_2. The result is

$$Y_2 = \frac{W}{b}\left[\left(\tfrac{1}{2}b - a\frac{v^2}{gr}\right)\cos H + \left(\tfrac{1}{2}b\frac{v^2}{gr} + a\right)\sin H\right] \qquad (3)$$

21. Car Moving Without Exerting Lateral Pressure.—The pressure X' exerted between the rail and the flange increases with the velocity v, as is evident from formula **1** of the preceding article. As this pressure is very injurious both to the rail and to the wheels, the track is so constructed as to eliminate it, that is, the superelevation e is so made that, for the greatest velocity of trains moving on the track, the pressure X' shall be zero. Putting the second member of formula **1** of the last article equal to zero,

$$\frac{v^2}{gr}\sqrt{b^2 - e^2} = e;$$

whence, squaring and solving for e,

$$e = \frac{b\dfrac{v^2}{gr}}{\sqrt{1 + \left(\dfrac{v^2}{gr}\right)^2}} \qquad (1)$$

Usually, $\left(\frac{v^2}{gr}\right)^2$ is a very small fraction. If this fraction is neglected, the following approximate formula is obtained:

$$e = b\frac{v^2}{gr} \qquad (2)$$

22. When X' is zero, the algebraic sum of the moments of Y_1 and Y_2 about G is zero, since the moments of both W and P are zero; and, as the lever arms of Y_1 and Y_2 are equal, it follows that $Y_1 = Y_2$. Also, since the resultant P of Y_1, Y_2, and W is horizontal, the sum of the vertical components of Y_1 and Y_2 must be numerically equal to W, and

the sum of their horizontal components must be equal to P. If, therefore, a horizontal line AQ is drawn through A, the vector QG will represent $Y_1 + Y_2$, and the vector QA will represent the horizontal component of $Y_1 + Y_2$, this component being equal to P. It is thus seen that, in this case, the centripetal force P is obtained by resolving the weight W into two components—one, GQ, perpendicular to the track, and one, GE, horizontal: GQ simply balances the pressure $Y_1 + Y_2$ of the rails, while GE is the centripetal force.

23. Car on the Point of Upsetting About the Outer Wheel.—If the velocity increases sufficiently, the pressure X' will become so great that the car will upset by turning about the rail R_1. When the car is on the point of upsetting, the wheel F_2 is on the point of being lifted from the rail R_2, and there is, consequently, no pressure between the wheel and the rail at R_2. The relation that exists between e and v for this condition is obtained by making $Y_2 = 0$ in formula **3** of Art. **20**, expressing cos H and sin H in terms of e and b, and solving for v or e. The process is comparatively complicated, and will not be given here. There is, however, an approximate solution that is sufficiently close for practical purposes. In railroad work, the angle H is always very small, and its cosine can, without any appreciable error, be taken equal to 1. Putting the second member of formula **3** of Art. **20**, equal to zero, and writing 1 for cos H and $\frac{e}{b}$ for sin H (see Fig. 11),

$$\tfrac{1}{2} b - a \frac{v^2}{gr} + \left(\tfrac{1}{2} b \frac{v^2}{gr} + a\right) \frac{e}{b} = 0$$

whence
$$\left(a - \tfrac{1}{2} e\right) \frac{v^2}{gr} = \tfrac{1}{2} b + \frac{ae}{b} = \frac{b^2 + 2ae}{2b}$$

and
$$v = \sqrt{\frac{gr(b^2 + 2ae)}{b(2a - e)}}$$

EXAMPLE 1.—A curved track is to be constructed for a maximum train velocity of 45 miles per hour. If the radius of the curve is 2,500 feet, and the distance between centers of rails 5 feet, what must

be the superelevation of the outer rail, that there may be no lateral pressure?

SOLUTION —Formula 2 of Art. **21** will be used Here, $b = 5$; $r = 2{,}500$, $g = 32\ 16$, and $v = \frac{45 \times 5{,}280}{60 \times 60} = 66$ ft per sec. Therefore,

$$e = 5 \times \frac{66^2}{32\ 16 \times 2{,}500} = .271 \text{ ft} = 3\tfrac{1}{4} \text{ in}, \text{ nearly} \quad \text{Ans.}$$

EXAMPLE 2 —With the track constructed as in example 1, what is the maximum velocity a train can have without upsetting, the height of the center of gravity of each car above the track being 6 feet?

SOLUTION —The formula in Art. **23** gives

$$v = \sqrt{\frac{32\ 16 \times 2{,}500\,(5^2 + 2 \times 6 \times .271)}{5\,(2 \times 6 - .271)}} = 196\ 80 \text{ ft per sec.}$$
$$= 134 \text{ mi. per hr., nearly.} \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE

1. A curved track of 1,000 feet radius is to be built for a maximum train velocity of 40 miles per hour, the distance between rails is 5 feet What must the superelevation be, that there may be no lateral pressure? Ans 535 ft

2 The radius of a curved track being 1,500 feet, the superelevation $3\frac{1}{4}$ inches, and the distance between the centers of rails 5 feet, what is the maximum velocity a train can have without exerting any lateral pressure? Ans. 36.17 mi. per hr.

WORK AND ENERGY

WORK

24. Definition of Work.—Let a force F, supposed to remain constant in magnitude and direction, act on a body while the point of application of the force undergoes a certain displacement. This displacement does not necessarily have the direction of the force, nor is the force F necessarily the only force acting on the body. Let the projection of the displacement on a line parallel to the direction of the force be denoted by s. Then, the product Fs is called the **work** of the force F for the given displacement.

Thus, in Fig. 12, if the point of application of the force F moves from A to A', its displacement is $A A'$; the projection of this displacement on a line parallel to the line of action

of the force is AS, $A'S$ being perpendicular to the direction of the force. In this case, then, $s = AS$, and the work of the force, while the point of application moves from A to A', is $F \times AS$. The projection AS is called the **component** of the displacement AA' parallel to the line of action of the force When, as in Fig. 12, the projection of A' on AS moves in the same direction as the force, *the force* is said to do work *on* the body on which it acts When, as in Fig 13, the projection of A' on AS moves in a direction opposite to that of the force, *the body* is said to do work *against the force,* or the force is said to do the negative work $-F \times AS$ on the body.

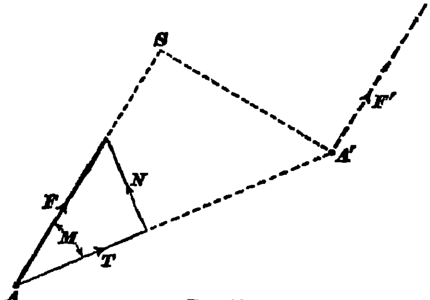


FIG. 12

If the line of action of the force F, Figs. 12 and 13, coincide with the path AA' of the point of application, then $s = AA'$. If the point of application moves in the direction of the force, the work of the force is $F \times AA'$; otherwise, $-F \times AA'$.

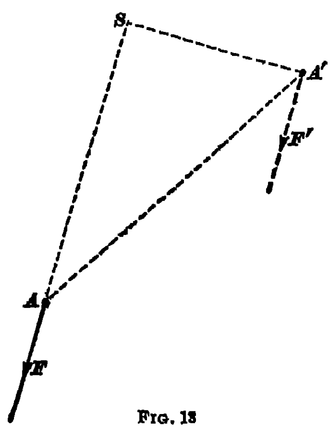


FIG. 13

If the line of action of F is perpendicular to AA', the projection AS reduces to the point A, and $F \times AS = F \times 0 = 0$; that is, the work done by the force is zero. This shows that, *when the point of application of a force moves in a direction perpendicular to that of the force, the force does no work.*

25. Formulas for Work.—Denoting the work of the force F, Fig. 12, by U, we have

$$U = F \times AS \qquad (1)$$

If F is resolved into two components T and N, the former along AA', and the latter perpendicular to AA', then,

$$T = F\cos M, \text{ and } F = \frac{T}{\cos M}$$

This value in (1) gives

$$U = \frac{T}{\cos M} \times AS = T \times \frac{AS}{\cos M} = T \times AA'$$

This formula shows that *the work of a force can be obtained by multiplying the component of the force in the direction of the displacement of its point of application, by the length of that displacement.*

26. Work of the Resultant of Several Forces. *When several forces act on a body, the algebraic sum of their works is equal to the work of their resultant.* This is an immediate consequence of the principle stated in the preceding article. Let F be the resultant of several forces, say F_1, F_2, F_3. Let the body on which these forces act move in any direction through a distance x, and let the corresponding work of the forces be U, U_1, U_2, U_3. Denoting the components of the forces in the direction of the displacement by X, X_1, X_2, X_3, we have, according to the preceding article, $U = Xx$, $U_1 = X_1 x$, $U_2 = X_2 x$, $U_3 = X_3 x$. Therefore,

$$U_1 + U_2 + U_3 = (X_1 + X_2 + X_3)x$$

According to the principles of statics, $X_1 + X_2 + X_3 = X$; therefore, $U_1 + U_2 + U_3 = Xx = U$, as stated. It should be remembered that the resultant F is a force that can replace the forces F_1, F_2, F_3, not a force acting simultaneously with them.

Since, when several forces are in equilibrium, their resultant is zero, it follows that, if a body is moving under the action of balanced forces (in which case the body must be moving with uniform velocity), the algebraic sum of the works of the forces for any displacement of the body is equal to zero.

27. Effort and Resistances.—Usually, forces are applied to bodies in order to produce or to prevent motion. The forces thus applied are called **efforts**, and those that

oppose them are called **resistances.** Thus, when a train is moved by a locomotive, the pull of the locomotive is the effort; the friction of the wheels and the resistance of the air are the resistances. Generally, when the work done on a body is referred to, the work done by the effort is meant, although, in some cases, its value is determined by calculating the work of the resistance. Thus, if a weight W is raised through a height h, the work of the resistance W is numerically Wh, which, if the body comes to rest at the end of the distance h, is also the work of the effort.

28. Unit of Work.—As work is measured by the product of a force and a distance, its numerical value depends on the units of force and length employed. The **unit of work** may be defined as the work done by a force equal to the unit of force acting through a distance equal to the unit of length. If the unit of force is the pound, and the unit of length is the foot, work is expressed in **foot-pounds;** if the unit of force is the ton, and the unit of length is the inch, work is expressed in **inch-tons;** etc.

29. Work Done in Raising a System of Bodies. It can be shown by the use of advanced mathematics that *the work performed in raising a system of bodies is equal to the aggregate weight of all the bodies multiplied by the height through which their center of gravity is raised.* Thus, if stones piled in any manner at the bottom of a tunnel shaft are raised and formed into a pile on the surface, the work performed is equal to the combined weight of all the stones multiplied by the vertical distance between the centers of gravity of the two piles. The stones may be raised one or more at a time and deposited on the surface in any manner whatever; the work will always be the same.

EXAMPLE 1.—The average pull of a locomotive on the train being 7 5 tons, what is the work done by the locomotive in hauling the train 1 mile?

SOLUTION.—Here F = 7 5 T., s = 1 mi. = 5,280 ft Therefore,
U = 7.5 × 5,280 = 39,600 ft.-tons. Ans.

EXAMPLE 2.—Given an inclined plane CD, Fig 14, 500 feet long and inclined to the horizontal at an angle H of 30°, it is required to determine the work necessary to just pull up from C to D a block B whose weight W is 250 pounds The coefficient of friction f between the block and the plane is .25

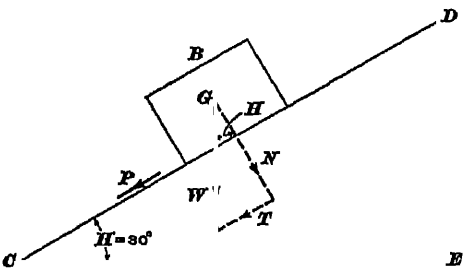


FIG. 14

SOLUTION —The resistances opposing the motion of the block are the weight W and the friction P The work U of the effort is numerically equal to the sum of the works of those resistances Resolving W into two components N and T, the former normal to the plane and the latter along the plane, we have (see *Analytic Statics*), $T = W \sin H$, $N = W \cos H$, $P = fN = fW \cos H$. The work done against P, in foot-pounds, is $P \times CD = 500\ P$, and that done against W is $T \times CD = 500\ T$. Therefore,

$$U = 500\ T + 500\ P = 500\ (T + P) = 500\ (W \sin H + fW \cos H)$$
$$= 500\ W (\sin H + f \cos H) = 500 \times 250\ (\sin 30° + .25 \times \cos 30°)$$
$$= 89{,}563 \text{ ft-lb. Ans.}$$

EXAMPLE 3.—A cylindrical cistern 10 feet in diameter is filled with water to a distance of 50 feet above the bottom. What work must be done in pumping all the water into a tank 15 feet in diameter, whose bottom is 100 feet above the bottom of the cistern? The weight of water is taken equal to 62.5 pounds per cubic foot.

SOLUTION.—The volume V of the water to be raised is

$$\left(\frac{\pi \times 10^2}{4} \times 50\right) \text{cu. ft.}$$

For the weight of this volume we have

$$W = V \times 62.5 = \frac{\pi \times 10^2}{4} \times 50 \times 62.5$$

If y is the height of the water in the tank after pumping,

$$V = \frac{\pi \times 15^2}{4} y$$

whence $$y = \frac{4\ V}{\pi \times 15^2} = \frac{4 \times \frac{\pi \times 10^2}{4} \times 50}{\pi \times 15^2} = \frac{10^2 \times 50}{15^2}$$

The distance of the center of gravity of the water in the cistern from the bottom of the cistern is 25 ft. The distance of the center of gravity of the water in the tank from the bottom of the tank is $\frac{y}{2}$.

For the distance h between the two centers of gravity we have, therefore,

$$h = 100 - 25 + \frac{y}{2} = 75 + \frac{10^2 \times 50}{2 \times 15^2}$$

Then (Art 29),

$$U = Wh = \frac{\pi \times 10^2}{4} \times 50 \times 62\ 5\left(75 + \frac{10^2 \times 50}{2 \times 15^2}\right) = 21{,}135{,}000 \text{ ft -lb}$$

Ans.

NOTE —In practical problems, it is seldom necessary to use more than five significant figures

POWER

30. Work, as defined in Art. **24**, has no reference to time. The work performed in raising a given weight through a given height is the same whether the time occupied is a second or an hour. However, when the capacities of different agents for doing work are to be compared, time must be considered; and to indicate the amount of work performed in a given time, or the rate of doing work, the term **power** is used. If an engine does in an hour twice as much work as another engine, it is said to have twice the power of the second engine.

31. Horsepower.—Power may be expressed in terms of any convenient units, as foot-pounds per second, foot-tons per minute, etc. Thus, if an engine working uniformly raises 40,000 pounds through a distance of 10 feet in 1 minute, the work it performs per minute is 400,000 foot-pounds, and its power may be stated as 400,000 foot-pounds per minute. The foot-pound per minute is too small a unit for many purposes, and a larger unit is more generally used by engineers. This unit is called the **horsepower**, and is equal to 33,000 foot-pounds per minute, or 550 foot-pounds per second. The abbreviation for horsepower is H. P.

Let F = effort (or resistance), in pounds;
s = distance, in feet, that point of application is moved;
t = time occupied, in minutes;
t_1 = time occupied, in seconds;
H = horsepower.

Then, $$H = \frac{Fs}{33{,}000\,t} = \frac{Fs}{550\,t_s}$$

EXAMPLE 1.—What horsepower is required to pull a train weighing 400 tons at a speed of a mile per minute, if the resistance at this speed is 12 pounds per ton?

SOLUTION.—The force F necessary to overcome the resistance is $400 \times 12 = 4{,}800$ lb. $s = 5{,}280$ ft. Substituting in the formula,

$$H = \frac{4{,}800 \times 5{,}280}{33{,}000 \times 1} = 768 \text{ H. P. Ans.}$$

EXAMPLE 2.—A crane hoists a load of 5 tons a height of 22 feet in 20 seconds. What horsepower is developed?

SOLUTION.—The resistance F is $5 \times T = 10{,}000$ lb.; 20 sec. $= \frac{1}{3}$ min. Substituting in the formula,

$$H = \frac{10{,}000 \times 22}{33{,}000 \times \frac{1}{3}} = 20 \text{ H. P. Ans.}$$

or, $$H = \frac{10{,}000 \times 22}{550 \times 20} = 20 \text{ H. P. Ans.}$$

ENERGY

32. Kinetic Energy.—Let a body of mass m and weight W be moving with a velocity v_0. If a force F is applied to the body in a direction opposite to the direction of motion, that force will bring the body to rest in a distance s, such that

$$Fs = \tfrac{1}{2}\, m\, v_0^2 \qquad (1)$$

This formula, which should be committed to memory, is obtained from the following, derived in *Fundamental Principles of Mechanics*:

$$F = \frac{m(v^2 - v_0^2)}{2\,s} \qquad (2)$$

In the present case, $v = 0$, and, therefore,

$$F = \frac{-m\,v_0^2}{2\,s},\ Fs = -\tfrac{1}{2}\,m\,v_0^2$$

The negative sign simply indicates that F and v have opposite directions. When only numerical values are considered, the sign is disregarded, and the equation written as in formula **1**.

Notice that Fs is the work done by the body against the force in the space s. Whatever the force may be, this work

is always the same, since it is always equal to $\frac{1}{2} m v_0^2$ It follows, then, that the body can, on account of its velocity and mass, perform an amount of work numerically equal to $\frac{1}{2} m v_0^2$.

33. The capacity that an agent has for performing work is called **energy.** The energy that a moving body has, on account of its velocity and mass, is called the **kinetic energy** of the body; and, as just explained, is measured by the product $\frac{1}{2} m v_0^2$. Since this quantity is equivalent to work, it is expressed in units of work, such as foot-pounds or foot-tons

Denoting the kinetic energy of a moving body by K, and replacing m by $\frac{W}{g}$ (see *Fundamental Principles of Mechanics*), formula **1** of Art. **32** may be written,

$$K = \frac{W v_0^2}{2g} \qquad (1)$$

Also,

$$Fs = \frac{W v_0^2}{2g} \qquad (2)$$

EXAMPLE —Find the kinetic energy, in foot-tons, of a car weighing 25,000 pounds, and moving at the rate of 30 miles an hour.

SOLUTION —Here, $W = 25{,}000$ lb $= \frac{25}{2}$ tons. The velocity of the train, in feet per second, is

$$\frac{30 \times 5{,}280}{60 \times 60} = 44 \ (= v)$$

Therefore (formula **1**),

$$K = \frac{1}{2} \times \frac{25}{2g} \times 44^2 = 376\ 24 \text{ ft -tons. Ans.}$$

NOTE —The velocity has been reduced to feet per second, because g is expressed in feet per second We might also find v in feet per hour and reduce g to feet per hour In all cases, it is necessary to refer v and g to the same units.

34. Equation of Energy.—From formula **2** of Art. **32**, we have, replacing m by $\frac{W}{g}$,

$$Fs = \frac{W}{2g}(v^2 - v_0^2) \qquad (1)$$

This formula gives the work performed by an unbalanced force in changing the velocity of a body from v_0 to v, or its

kinetic energy from $\frac{W}{2g} v_0^2$ to $\frac{W}{2g} v^2$. If the force F is the difference between an effort P and a resistance R, then,

$$(P - R)\, s = \frac{W}{2g}(v^2 - v_0^2);$$

whence
$$Ps = Rs + \frac{W}{2g}(v^2 - v_0^2) \qquad (2)$$

This formula is called the **equation of energy**, and may be stated thus:

The work done by the effort is equal to the work done against the resistance plus the increase of kinetic energy.

EXAMPLE —In hoisting coal from a mine, the load to be hoisted, including cage and car, is 12,000 pounds, the load starts from rest, and when it is 50 feet from the bottom, it is moving with a velocity of 30 feet per second What is the pull in the hoisting rope?

SOLUTION.—The resistance R is the weight, 12,000 lb. The effort P is the unknown pull in the rope. $v_0 = 0$, $v = 30$ ft per sec., and $s = 50$ ft Substituting in the formula,

$$Ps = Rs + \frac{W}{2g}(v^2 - v_0^2) = 12{,}000 \times 50 + \frac{12{,}000}{2 \times 32.16}(30^2 - 0^2)$$
$$= 767{,}910 \text{ ft.-lb.}$$
$$P = \frac{767{,}910}{50} = 15{,}358\ 2 \text{ lb. Ans.}$$

NOTE —In the solution of this example it is assumed that, when the load has reached a height of 50 feet, it is still being accelerated at the average rate of acceleration necessary to give it a velocity of 30 feet per second in a space of 50 feet When the velocity becomes uniform, the pull in the rope is that due to the load only, since there is then no increase in the kinetic energy.

35. Potential Energy.—As has been explained, a body may have a capacity for doing work by reason of its velocity. There are, however, other states or conditions than that of motion that give a body a capacity for work: (1) Suppose that a body of weight W is raised through a height h; in thus raising the body, the work Wh is done against the force W. If, now, the body is permitted to descend through the same vertical distance h, the weight W becomes the acting force, and the work Wh is done by it. The body in its highest position has the work Wh stored in it, and thus possesses a stock of energy equal also to Wh. (2) Suppose that a spring is extended or compressed; work is done against the resistance of the spring to change of length.

If the spring is released, it will return to its original form, and in so doing can do precisely the amount of work that was expended on it. Thus, in the extended or compressed state, there is an amount of work stored in the spring, and the spring possesses energy because of its stretched or compressed condition. Similarly, compressed air possesses energy merely because it is compressed and can do work in returning to its original state.

The energy that a body possesses by reason of its *position*, *state*, or *condition* is called **potential energy.**

36. It is to be remarked that the potential energy that a body has, due to its position, is relative to a certain plane below the body. Thus, if the body weighs W and its height above the ground is h, its potential energy with respect to the ground is Wh With respect to a plane whose distance below the body is h_1, the potential energy is Wh_1

37. Conservation of Energy.—The law of the conservation of energy asserts that energy cannot be destroyed. When energy apparently disappears, it is found that an equal amount appears somewhere, though perhaps in another form. Frequently, potential energy changes to kinetic energy or vice versa. To illustrate, take the case of a steam hammer or a pile driver. The ram is raised to a height h above the pile, and, on being released, strikes the head of the pile and comes to rest. In its highest position, the ram has the potential energy Wh In falling, it attains a velocity v that, just at the instant of striking, has the magnitude $\sqrt{2gh}$, hence, at this instant, the ram has lost all its potential energy, but has a kinetic energy $\frac{Wv^2}{2g} = Wh.$ The loss of one kind is therefore just balanced by the gain of the other kind. After the blow, the ram comes to rest and has neither potential nor kinetic energy. Apparently there is a loss of energy, but really the energy has for the most part been expended in doing the work of driving the pile a certain distance into the earth. A small part has

been expended in heating the ram and head of the pile, and another part in producing sound.

Heat is a form of energy due to the rapid vibration of the molecules of bodies. In many cases where energy apparently disappears, it reappears in the form of heat Thus, the work done against frictional resistances appears as heat, which is usually dissipated into the air as fast as produced

Conversely, there are examples of heat energy transformed into other forms. A pound of coal has potential energy, which, when the coal is burned, is liberated and changed into heat energy. The heat applied to water produces steam, and the steam has potential energy. Finally, the steam in giving up its energy does work in a steam engine, and this work is expended in overcoming the friction of shafts, belts, and machine parts, and ultimately reappears in the form of heat. However, in all these transformations, the total amount of energy—the sum of the kinetic and potential energy—remains always the same. Such is the law of the **conservation of energy.**

EFFICIENCY

38. In every machine, there is an effort or driving force that produces motion in the machine, and a resistance against which the machine does work. Take, for example, the raising of a load by a crane. The effort is exerted by the workman on the crank of the windlass, and the resistance is the weight of the load lifted. In the case of a machine tool for cutting metal, the effort is the pull of the belt on the driving pulley, and the resistance is the resistance of the metal to the cutting tool. In a given time, say 1 minute, a quantity of work is done by the effort and another quantity of work is done against the resistance. Were it not for frictional resistances, these two works would be just equal. In actual cases, however, only a part of the work of the effort is usefully expended in doing work against the resistance. The remainder is used in overcoming the friction between the various sliding surfaces, journals and bearings, etc. The work thus

expended appears in the form of heat energy, and serves no useful purpose.

The ratio $\frac{\text{useful work}}{\text{total work of effort}}$ or $\frac{\text{useful work}}{\text{total energy supplied}}$ is the **efficiency** of the machine.

Representing the efficiency by E, the useful work by U_o, and the total work (= energy supplied) by U, we have,

$$E = \frac{U_o}{U}$$

EXAMPLE.—A water motor receives 150 cubic feet of water per second, with a velocity of 45 feet per second. If the maximum power that can be obtained from the motor is 459 H. P., what is its efficiency?

SOLUTION.—The weight of the water delivered per second is 150×62.5 lb., the kinetic energy of which (Art. **33**) is

$$\frac{150 \times 62.5 \times 45^2}{2g} \text{ ft.-lb. per sec. } (= U)$$

Since 1 H. P. = 550 ft.-lb. per sec., the useful work per second done by the motor is (459×550) ft.-lb. $(= U_o)$. Therefore,

$$E = \frac{459 \times 550}{\frac{150 \times 62.5 \times 45^2}{2g}} = .855, \text{ or } 85.5 \text{ per cent. Ans.}$$

HYDROSTATICS

DEFINITIONS—PROPERTIES OF LIQUIDS

1. Hydromechanics is that branch of mechanics that deals with liquids, their properties, and their applications to engineering. It may be divided into two chief branches: *hydrostatics* and *hydraulics*.

2. Hydrostatics treats of liquids in a state of rest.

3. Hydraulics treats of liquids in motion.

4. In the first of these divisions are included such problems as the pressure of water on enclosing vessels and submerged surfaces; in the second are included problems relating to the flow of liquids through orifices, in pipes, and in channels.

5. Liquid Bodies.—A **liquid body,** or simply a **liquid,** is a body whose molecules change their relative positions easily, being, however, held in such a state of aggregation that, although the body can freely change its shape, it retains a definite and invariable volume, provided the pressure and temperature are not changed. Water and alcohol are examples of liquid bodies.

6. A **perfect liquid** is a liquid without internal friction; that is, one whose particles can move on one another with absolute freedom. On account of this characteristic property, a perfect liquid offers no resistance to a change of form.

7. A **viscous liquid** is a liquid that offers resistance to rapid change of form on account of internal friction, or **viscosity.** Tar, molasses, and glycerine are examples of viscous liquids.

All liquids are more or less viscous. For the purposes of hydrostatics, however, water, which is the liquid mainly dealt with in this work, may be treated as a perfect liquid, its viscosity at ordinary temperatures being too small to be taken into account.

8. Compressibility.—All liquids offer great resistance to change in volume; that is, they can be compressed but little. Under the pressure of one atmosphere (about 14.7 pounds per square inch), water is compressed about $\frac{1}{20000}$ of its original volume. For engineering purposes, it may be assumed that water is practically incompressible.

9. Density.—The **density** of a homogeneous body is the mass of the body per unit of volume, and may be obtained by dividing the total mass of the body by its volume.

If V, W, m, and D are, respectively, the volume, weight, mass, and density of a body, then,

$$D = \frac{m}{V};$$

or, since $m = \frac{W}{g}$ (see *Fundamental Principles of Mechanics*),

$$D = \frac{W}{Vg}$$

Both from the definition and from this formula, it follows that the density of any one body varies inversely as the volume of the body; in other words, when the mass of the body remains the same, the density is greater the less the volume of the body, or the space that the body occupies. This agrees with the ordinary use of the words density and dense, which are employed to denote the degree of compactness, a body being said to be more or less dense according as its molecules are supposed to be more or less close to one another.

10. Weight of Water.—The weight of 1 cubic foot of water varies with the temperature: at 39.2° F., which is the temperature of maximum density, it is 62.425 pounds. For nearly all engineering purposes, 62.5 pounds is used as a

convenient and sufficiently approximate value. This value will be used throughout this work.

Since a column of water 1 square inch in cross-section and 1 foot high is $\frac{1}{144}$ cubic foot, its weight is $62.5 \div 144 = .434$ pound. This value is of very frequent application, and should be memorized.

LIQUID PRESSURE

PASCAL'S LAW AND ITS APPLICATIONS

11. Statement of Pascal's Law.—*A perfect liquid transmits pressure equally in all directions.* This principle is **Pascal's law**, and follows directly from the definition of a perfect liquid.

The difference between a liquid and a solid as regards transmission of pressure may be illustrated as follows: In Fig. 1 are shown two cylindrical vessels of the same size. The vessel a is fitted loosely with a wooden block of the same size as the vessel. The vessel b is filled with water, whose depth is the same as the length of the wooden block in a. Both vessels are fitted with air-tight pistons P. For convenience, let the weight of the block and that of the water be neglected, and suppose that a force of 100 pounds is applied to each piston. Assume the piston area to be 10 square inches; then the pressure per square inch is $100 \div 10 = 10$ pounds. In the vessel a, this pressure will be transmitted, undiminished, to the bottom of the vessel; it is easy to see that there will be no pressure on the sides. In the vessel b, an entirely different result is obtained. The pressure on the bottom will be the same as in the other

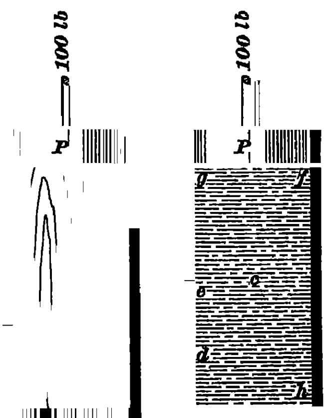


FIG. 1

case, that is, 10 pounds per square inch; but, owing to the fact that the molecules of water are perfectly free to move, this pressure of 10 pounds per square inch is transmitted in every direction with the same intensity; that is to say, the pressure at all points, such as *c*, *d*, *e*, *f*, *g*, *h*, due to the external force of 100 pounds, is exactly the same, namely, 10 pounds per square inch.

12. Verification of Pascal's Law.—An experimental proof of Pascal's law may be effected with the apparatus shown in Fig. 2 The vessel is filled with water, and is fitted with pistons of different diameters. Let a force be applied to the small piston *c*, as a result, the water in contact with *c* is subjected to a certain pressure According to Pascal's law, this pressure must be transmitted with undiminished intensity in all directions. For the sake of convenience, let the area of the piston *c* be 1 square inch, and let a force of 5 pounds be applied to it. The pressure exerted by the piston on the water is, therefore, 5 pounds per square inch. If the area of the piston *a* is 40 square inches, and Pascal's law is true, the water must exert on the face of *a* a total pressure of $40 \times 5 = 200$ pounds. It is found by experiment that a force of 200 pounds must be applied to the piston *a* to prevent it from moving outwards. This shows that the fluid pressure against *a* is 200 pounds, or 5 pounds per square inch, which is the same as that against *c*. Similar results are obtained with the pistons *b*, *d*, *e*, and *f*. If their areas are, respectively, 7, 6, 8, and 4 square inches, the pressures exerted on them are found to be, respectively, 35, 30, 40, and 20 pounds.

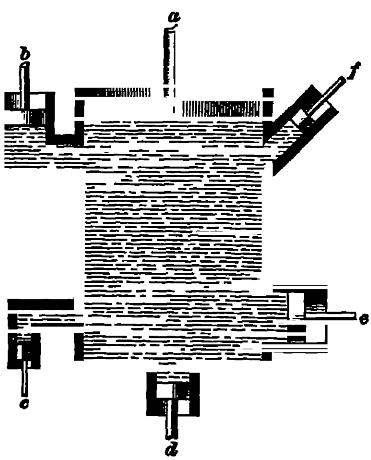


FIG 2

Pascal's law may be formally stated as follows: *The pressure per unit of area exerted anywhere on a mass of liquid is transmitted undiminished in all directions, and any surface in contact with the liquid will be subjected to this pressure in a direction at right angles to the surface.*

13. Application of Pascal's Law.—In Fig. 3, let the area of the piston a be 1 square inch, and that of b, 40 square inches. According to Pascal's law, 1 pound placed on a will balance 40 pounds placed on b. If a moves downwards 10 inches, then 10 cubic inches of water will be forced into the tube b. This will be distributed in the tube b in the form of a cylinder whose cubical contents must be 10 cubic inches, whose base has an area of 40 square inches, and whose altitude must therefore be $10 \div 40 = \frac{1}{4}$ inch. This shows that a movement of 10 inches of the piston a causes a movement of $\frac{1}{4}$ inch of the piston b. This is an example of the familiar principle of work that *the force, or effort, multiplied by the distance through which it moves, is equal to the resistance multiplied by the distance through which it moves.*

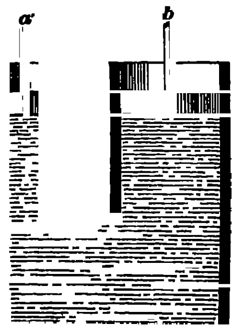


FIG 3

14. The Hydraulic Press.—The principle just stated finds an important application in the **hydraulic press**, which is shown in Fig. 4. As the lever a is depressed, the piston b is forced down on the water in the cylinder c. The water is forced through the bent tube d into a cylinder fitted with a large piston e, and causes that piston to rise, the platform f is thus lifted, and the bales placed between it and an upper fixed platform are compressed. Assume the area of the piston b to be 1 square inch, and that of the piston e to be 100 square inches. Also, assume the length of the lever between the hand and the fulcrum to be ten times the length between the fulcrum and the piston rod b. If the end of the lever is depressed, say 10 inches, the piston b is

depressed one-tenth of 10 inches, or 1 inch, and the piston *e* is raised $\frac{1}{100}$ inch, since 1 cubic inch of water is displaced. If P denotes the force applied and Q the pressure on the platform f, then $P \times 10$ inches $= Q \times \frac{1}{100}$ inch, hence, $Q = 1{,}000\ P$, or $P = \frac{1}{1000}\ Q$. A force of 40 pounds applied

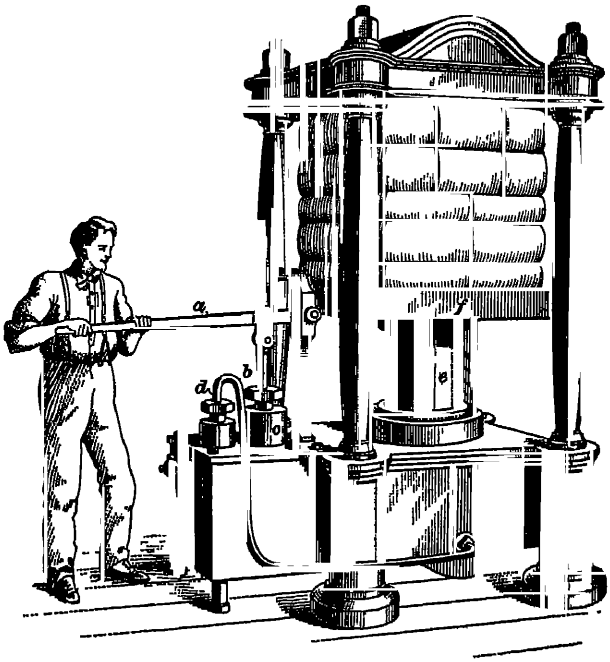


Fig 4

by hand thus produces a pressure of $40 \times 1{,}000 = 40{,}000$ pounds. But, if the average movement of the hand per stroke is 10 inches, it will require $1{,}000 \div 10 = 100$ strokes to raise the platform 1 inch, which again shows that what is gained in power is lost in speed.

GENERAL THEORY OF LIQUID PRESSURE

15. Downward Pressure.—The weight of a mass of liquid is a force, and will produce a pressure on any surface in contact with the liquid independently of the pressure produced by other forces. Thus, in the case of the vessel shown in Fig. 2, if the vessel is vertical so that the piston *d* is at the bottom, *d* is subjected, in addition to the pressure of 5 pounds per square inch transmitted from the piston *c*, to an additional pressure due to the weight of the water above it. Evidently, the piston *e* will be under an additional pressure due to the weight of the water, but the pistons *a*, *b*, and *f*, being above the liquid, will be subjected only to transmitted pressure.

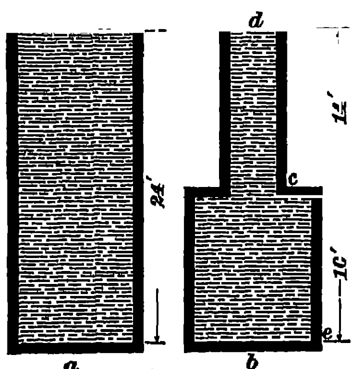


FIG 5

To arrive at the laws governing the pressure due to the weight of a liquid, let us consider first the pressure on the bottom of a vessel containing the liquid. In Fig. 5 are shown two vessels of different shape but with the water level in each at the same height—24 feet. Assume the area of the bottom of *a* to be 500 square inches, or $\frac{500}{144}$ square feet; then the volume of the water contained in *a* is $\frac{500}{144} \times 24 = 83\frac{1}{3}$ cubic feet, and the weight is $83\frac{1}{3} \times 62.5 = 5{,}208.3$ pounds. The pressure per square inch is $5{,}208.3 \div 500 = 10.42$ pounds, nearly.

An easier solution, however, is effected as follows: The weight of a column of water 1 square inch in cross-section and 1 foot long is .434 pound (Art. 10). Now, above each square inch of the bottom there is a column of water 24 feet high, whose weight is, therefore, $24 \times .434 = 10.42$ pounds, and this weight is the pressure per square inch on the bottom of the vessel.

In vessel *b*, assume the cross-section of the lower part to be 500 square inches and that of the upper part to be

100 square inches. It is evident that the volume of water contained is smaller than before, and it might be concluded that the pressure on the bottom is, consequently, smaller; but this conclusion would not be true, as will appear from the following reasoning: Let water be poured into the empty vessel until the lower part is filled; the pressure on the bottom is, according to the method of calculation just given, $10 \times .434 = 4.34$ pounds per square inch. Imagine, now, a thin piston or disk fitted tightly into the smaller part and resting on the surface of the water below, and then let water be poured in until the smaller part is full. This piston acts as the bottom of a vessel 14 feet high and with a uniform cross-section of 100 square inches. Hence, the pressure on this piston, due to the weight of the water above, is $14 \times .434 = 6.08$ pounds per square inch. According to Pascal's law, this pressure is transmitted in all directions, and therefore acts on the bottom of the vessel; hence, the total pressure on the bottom is $4.34 + 6.08 = 10.42$ pounds per square inch, the same as in the first case. It is evident that the result will be the same if the piston is left out.

16. From the reasoning used in the preceding paragraph, it appears that the pressure per unit of area on the bottom of a vessel depends only on the vertical distance between the bottom and the liquid surface, and not at all on the shape of the vessel. This principle is one of great importance.

17. Intensity of Pressure.—When a surface is subjected to a uniformly distributed pressure, the pressure per unit of area is called the **intensity of pressure**, or **unit pressure.** Usually, the unit of force is the pound, and that of area is either the square inch or the square foot; and so the intensity of pressure is expressed either in pounds per square inch or in pounds per square foot.

In general, if P is the total pressure uniformly distributed over a surface, and A is the area of the surface, the intensity of pressure p is given by the formula

$$p = \frac{P}{A}$$

18. Head.—The distance from any horizontal layer of a liquid body to the surface of the liquid is termed the **head** on that layer.

Let h = head, in feet, on any horizontal layer;
p = pressure per square inch on the layer, in pounds;
w = weight of a column of liquid 1 foot long and 1 square inch in cross-section.

Then,
$$p = w\,h \qquad (1)$$

For water, $w = .434$, and, therefore,
$$p = .434\,h \qquad (2)$$

Also, $h = \frac{p}{.434}$; that is,
$$h = 2.304\,p \qquad (3)$$

EXAMPLE 1.—The depth of water in a stand pipe is 80 feet (*a*) What is the pressure per square inch on the bottom? (*b*) What is the pressure per square inch on a horizontal layer 65 feet from the surface?

SOLUTION.—(*a*) Substituting the value of h in formula **2**,
$p = .434 \times 80 = 34.72$ lb per sq in. Ans
(*b*) Here $h = 65$, hence, substituting in formula **2**,
$p = .434 \times 65 = 28.21$ lb. per sq. in. Ans

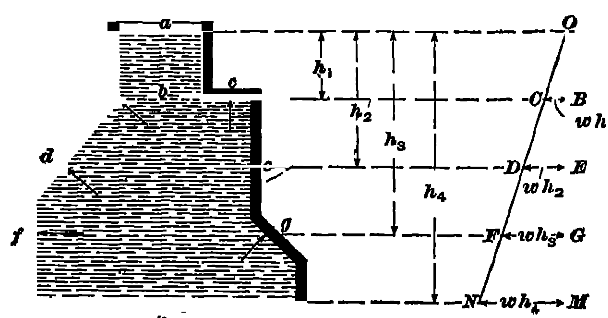
EXAMPLE 2.—What must be the height of water in a stand pipe to give a pressure of 80 pounds per square inch on the bottom?

SOLUTION.—Substituting in formula **3**,
$h = 2.304 \times 80 = 184.32$ ft. Ans.

19. Upward and Lateral Pressure.—So far, only downward pressure has been discussed. It is necessary now to consider upward pressure, as well as lateral (sidewise) pressure on vertical surfaces. Let the vessel shown in Fig. 6 (*a*) be filled with a liquid to the level *a*. The part of the liquid in *a b* rests on the layer at *b*, and produces over that surface an intensity of pressure of $w\,h_1$ pounds per square inch, where h_1 is the head, in feet, on the layer at *b*. According to Pascal's law, this intensity of pressure is transmitted to all the bounding surfaces below the level *b*; hence, there is a pressure of $w\,h_1$ pounds per square inch, due to the liquid in *a b*, exerted at *c*, *d*, *e*, *f*, *g*, and *k*, at right angles to

the surfaces. At any point below the level b there is, however, additional pressure due to the weight of the liquid between the point and the level b. At c, which is at the same level as b, the total upward pressure per unit of area is $w\,h_1$, the same as the downward pressure on the layer at b.

Consider now the layer at $d\,e$ at a distance h_2 below the surface a. The pressure on this layer, due to the weight of the liquid above, is $w\,h_2$ pounds per square inch, and by Pascal's law this pressure is transmitted to all parts of the bounding surface below the level $d\,e$, just as if the layer $d\,e$ were a solid piston. The liquid below $d\,e$ can exert no pressure at the points d and e; hence, at these points the pressure per unit of area is the same as the downward pressure on the



(a) FIG. 6 (b)

layer $d\,e$, namely, $w\,h_2$ pounds per square inch. The same reasoning shows that the lateral pressure per unit of area at the points f and g is $w\,h_3$, where h_3 is the head on the layer $f\,g$.

The following important law, which is a direct consequence of that of Pascal, may now be stated:

The intensity of pressure at any point of a surface enclosing a liquid is normal to that surface; it depends only on the depth of the point below the surface of the liquid, and is equal to the intensity of downward pressure on a horizontal layer of the liquid having the same head as the point in question.

The formula $p = w\,h$ is, therefore, general, and expresses the intensity of pressure at any point whose distance from the surface of the liquid is h.

EXAMPLE —In Fig 6, suppose the depth of the various layers below the level *a* to be as follows: depth of *b*, 10 feet, of *de*, 17 feet, of *fg*, 25 feet, and of *k*, 30 feet. The liquid being water, what are the pressures per square inch at the points *c*, *d*, *e*, *f*, *g*, *k*?

SOLUTION —Using formula **2** of Art **18** in each case, the pressures at the various points are found to be as follows

$$\text{Pressure at}\begin{cases} c = .434 \times 10 = 4.34 \text{ lb. per sq. in. Ans.} \\ d = .434 \times 17 = 7.38 \text{ lb. per sq. in. Ans.} \\ e = .434 \times 17 = 7.38 \text{ lb. per sq. in. Ans.} \\ f = .434 \times 25 = 10.85 \text{ lb. per sq. in. Ans.} \\ g = .434 \times 25 = 10.85 \text{ lb. per sq. in. Ans.} \\ k = .434 \times 30 = 13.02 \text{ lb. per sq. in. Ans.} \end{cases}$$

20. Graphic Determination of Pressure.—The varying intensity of pressure for different points below the surface of a liquid may be determined graphically as follows: Let OM, Fig. 6 (*b*), represent, to any convenient scale, the head h_4 on the bottom k of the vessel. Draw MN perpendicular to MO, and representing, to any convenient scale, the intensity of pressure $w h_4$ on the bottom of the vessel. It should be observed that OM represents a distance and MN a pressure, and that there is no necessary relation between the two scales. For instance, OM may represent h_4 to a scale of 2 feet to the inch; and MN, the pressure $w h_4$ to a scale of $\frac{1}{60}$ inch to the pound. Draw ON. Then, any horizontal line limited by OM and ON will represent the intensity of pressure at any depth equal to the vertical distance of that line below O. Thus, CB, whose distance below O is h_1, represents $w h_1$. Likewise, DE represents $w h_2$. If the pressure scale is $\frac{1}{60}$ inch to the pound, it will be found that the line FG measures about $\frac{11}{60}$, which represents 11 pounds per square inch. For very accurate work, the scale should be as large as possible. It should be observed that the tangent of the angle MON is numerically equal to w, since

$$\tan MON = \frac{MN}{MO} = \frac{w h_4}{h_4} = w$$

21. Pressure Due to External Load.—If the surface of a liquid is subjected to a pressure, this pressure, according to Pascal's law, is transmitted undiminished to all parts

f the enclosing vessel, and must be added to the pressure ue to the weight of the liquid. Assume the surface a of 'ig. 6 to be subjected to a pressure of 30 pounds per square ich. In the example of Art. **19**, the pressure at c, due to the ead of water, is 4.34 pounds per square inch. Adding the) pounds per square inch due to the external force applied t a, the result is $4.34 + 30 = 34.34$ pounds per square inch ikewise, the pressure at d and e is $7.38 + 30 = 37.38$ pounds er square inch; the pressure at f and g is $10.85 + 30 = 40.85$ ounds per square inch; and the pressure at k is $13.02 + 30$ $= 43.02$ pounds per square inch.

Let G = total load on surface of liquid, in pounds;

A = area of surface loaded, in square inches;

$p_0 = \frac{G}{A}$ = intensity of pressure on surface, in pounds per square inch;

p = intensity of pressure, in pounds per square inch, at a point h feet below liquid surface.

Then,
$$p = w\,h + p_0 = w\,h + \frac{G}{A} \qquad \textbf{(1)}$$

When the liquid is water,
$$p = .434\,h + p_0 = .434\,h + \frac{G}{A} \qquad \textbf{(2)}$$

EXAMPLE —A vessel filled with salt water weighing 1 03 times as uch as fresh water has a circular bottom 13 inches in diameter The p of the vessel is fitted with a piston 3 inches in diameter, on which laid a weight of 75 pounds. What is the intensity of pressure on the ttom, if the depth of the vessel is 18 inches?

SOLUTION —In this case, $w = 1.03 \times .434 = .447$ lb.; $h = 18$ in. 1.5 ft, and $A = 3^2 \times .7854$ sq. in By formula **1**,

$$p = .447 \times 1.5 + \frac{75}{3^2 \times .7854} = 11.28 \text{ lb per sq. in.} \quad \text{Ans.}$$

22. Atmospheric Pressure.—When a liquid is exposed the air, as in the cases represented in Figs. 5 and 6, there an external pressure, due to the weight of the air, on the rface of the liquid. This pressure is called **atmospheric essure.** It varies according to locality and atmospheric nditions, but, for nearly all practical purposes, it may taken as 14.7 pounds per square inch. This pressure is

transmitted by the liquid and should be treated as the pressure p_0 considered in the preceding article. It is customary, however, in hydrostatic and hydraulic calculations to take the atmospheric pressure as the zero of reference. When the pressure of a liquid is referred to, the pressure in excess of the atmospheric pressure is meant. This is called **gauge pressure.** When the atmospheric pressure is added to the gauge pressure, the resultant pressure, which is really the total pressure on the liquid or surface considered, is called **absolute pressure.** Thus, a gauge pressure of 20 pounds per square inch is equivalent to an absolute pressure of $20 + 14.7 = 34.7$ pounds per square inch.

23. Equilibrium of Liquids at Rest.—Since the pressure on a horizontal layer due to the weight of a liquid

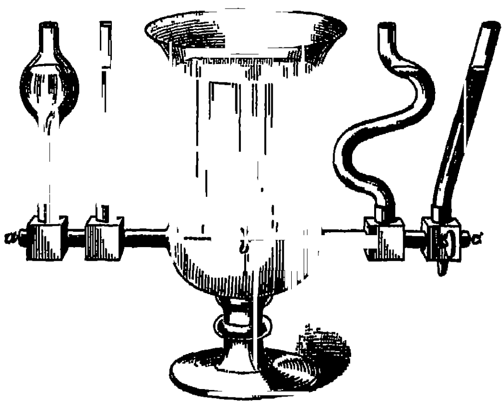


FIG 7

is dependent only on the height of the liquid, and not on the shape of the vessel, it follows that, if a vessel has a number of radiating tubes, as in Fig. 7, the water in each tube will be at the same level, no matter what may be the shape of the tubes. For, if the water is higher in one tube than in the others, the downward pressure at the level b, due to the height of water in this tube, is greater than that due to the height of the water in the other tubes. This excess of pressure will cause a flow toward the other tubes, which will continue until there is no further excess—that is, until the

free surfaces are at the same level. Then the liquid will come to rest, and will be in equilibrium. The principle here stated is embodied in the familiar saying, *Water seeks its level.* This principle explains why city reservoirs are located on high elevations, and why water, when leaving the hose nozzles, spouts so high. If there were no resistance by friction and air, the water would rise to a height equal to the level of the reservoir If a long pipe with a length equal to the vertical distance between the nozzle and the level of the water in the reservoir were attached to the nozzle and held vertically, the water would just reach the end of the pipe. If the pipe were lowered slightly, the water would trickle out. Fountains, canal locks, and artesian wells are examples of the application of this principle.

EXAMPLE 1 —The water in a city reservoir is 150 feet above a hydrant What is the pressure per square inch at the hydrant?

SOLUTION —By formula **2**, Art **18**,

$p = 434 \times 150 = 65\ 1$ lb per sq in Ans

EXAMPLE 2 —The pressure on a water main, when the water is not flowing, is shown by a gauge to be 72 pounds per square inch What is the elevation of the reservoir above the main?

SOLUTION.—By formula **3**, Art **18**,

$h = 2.304 \times 72 = 165.89$ ft Ans.

EXAMPLES FOR PRACTICE

1. What is the intensity of pressure on the bottom of a stand pipe 90 feet high? Ans 39 06 lb. per sq in.

2. A cylindrical vessel 15 inches in diameter is filled with water. The top of the vessel is fitted with a piston on which is laid a weight of 300 pounds. The depth of the vessel being 24 inches, determine the intensity of pressure on the bottom of the vessel

Ans 2 566 lb per sq. in.

3. If the intensity of pressure, due to an external load, on a vessel filled with water is 30 pounds per square inch, what is the intensity of pressure at a point 12 inches below the surface of the water?

Ans. 30.434 lb. per sq. in

PRESSURE ON AN IMMERSED SURFACE

24. Total Pressure on a Flat Immersed Surface. Let MN, Fig. 8, be a plate immersed in water. It is proposed to determine the total pressure acting on the upper surface of this plate. In the first place, the total pressure P, being the resultant of the pressures acting at all the points of MN, is perpendicular to MN. The magnitude of P is determined by means of the following principle, which is derived by advanced mathematics:

The total pressure on any plane surface immersed in a liquid is equal to the product of the following quantities (1) the area of the surface, (2) the distance of the center of gravity of the surface from the level of the liquid, (3) the weight of the liquid per unit of volume.

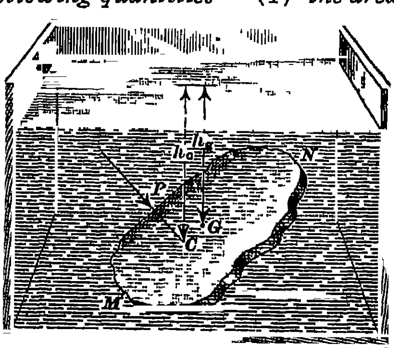


FIG. 8

In applying this principle, length, area, and volume should be referred to the same unit. Thus, if distances are expressed in feet, areas should be expressed in square feet and volumes in cubic feet. The principle obviously applies to the pressure on the bounding surfaces of the vessel or receptacle containing the liquid as well as to the pressure on the immersed surface.

Let A be the area of the surface MN, Fig 8; G, the center of gravity of that surface, its distance from the level of the liquid being h_g; and W, the weight of the liquid per unit of volume. Then, the foregoing principle may be stated in symbols thus:

$$P = A h_g W$$

It should be particularly noticed that the total pressure P does not act through the center of gravity G. The determination of the point of application of P will be dealt with farther on.

EXAMPLE.—To find the total pressure on the side $MLNK$ of the rectangular tank shown in Fig. 9, the dimensions being as indicated.

SOLUTION.—Since $MLNK$ is a rectangle, its center of gravity G is at a distance from the upper side LN equal to $\frac{KN}{2}$, or 6.25 ft. ÷ 2 = 3.125 ft.; hence, h_g = 3.125 ft. Also, $A = MK \times KN = 3.2 \times 6.25$, and W = 62.5 lb.

Substituting these values in the formula for P,

$$P = 3.2 \times 6.25 \times 3.125 \times 62.5 = 3{,}906 \text{ lb., nearly} \quad \text{Ans.}$$

25. Center of Pressure.—The point where the line of action of the total or resultant pressure acting on an immersed surface meets that surface is called the **center of pressure.** This point does not coincide with the center of gravity of the surface unless the latter is horizontal. In all other cases, the center of pressure lies below the center of gravity. Thus, in Fig. 8, the center of pressure C lies below the center of gravity G. A general formula for determining the position of the center of pressure cannot be either derived or applied without the use of advanced mathematics. Special formulas applying to some important cases are given in the following articles. The distance of the center of pressure from the level of the liquid will in all cases be denoted by h_c, as shown in Fig. 8.

FIG. 9

26. Center of Pressure of a Rectangular Surface. In Fig. 10 is represented a vessel V containing a liquid. The bounding surface $RSTU$ is shown inclined to the vertical; but the formulas derived in this and the following two articles apply both to an inclined and to a vertical surface. Let $KLMN$ be a rectangular part of the inclined surface, the sides KL and MN being horizontal. The line EFY is an axis passing through the middle points of KL and MN. The heads on the points E and F are denoted by h_1 and h_2, respectively. It should be observed that the formulas given

below apply whether the rectangular surface considered forms part of one of the bounding surfaces of the vessel or whether it is the surface of any immersed body.

The center of pressure C is on the line EF, and its depth h_c below the level of the liquid is given by the formula

$$h_c = \frac{2}{3}\left(\frac{h_1^3 - h_2^3}{h_1^2 - h_2^2}\right) \qquad (1)$$

To find the distance FC, draw FHQ perpendicular to the

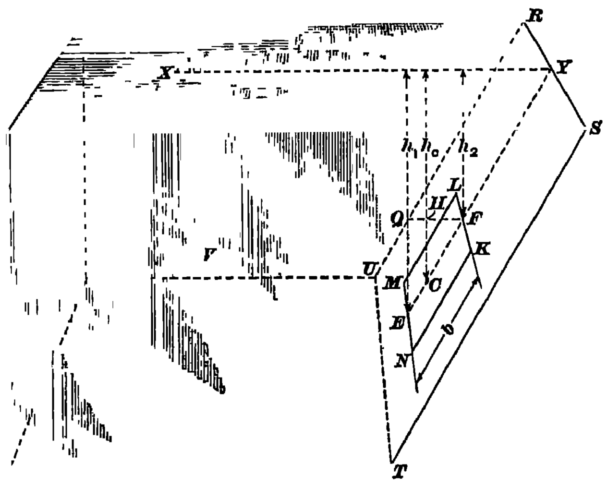


FIG 10

verticals through F, C, and E, and denote the length EF, or NK, by b. The similar triangles FCH and FEQ give $\frac{FC}{FE} = \frac{CH}{EQ}$; that is, $\frac{FC}{b} = \frac{h_c - h_2}{h_1 - h_2}$; whence, the following equation obtains:

$$FC = \frac{h_c - h_2}{h_1 - h_2} \times b \qquad (2)$$

If the value of h_c found by formula **1** is substituted in formula **2**, the result, after several transformations have been made, is

$$FC = \frac{2h_1 + h_2}{3(h_1 + h_2)} \times b \qquad (3)$$

27. If the edge KL is flush with the surface of the water, $h_2 = 0$, and formulas **1** and **3** of the preceding article become, respectively,

$$h_c = \tfrac{2}{3} \times \frac{h_1^3}{h_1^2} = \tfrac{2}{3} h_1 \qquad (1)$$

$$FC = \frac{2\,h_1}{3\,h_1} \times b = \tfrac{2}{3} b \qquad (2)$$

28. Center of Pressure of Triangular Surface. Only the case of an isosceles triangle with its base horizontal will be treated in this Section. And first, a triangle MFN,

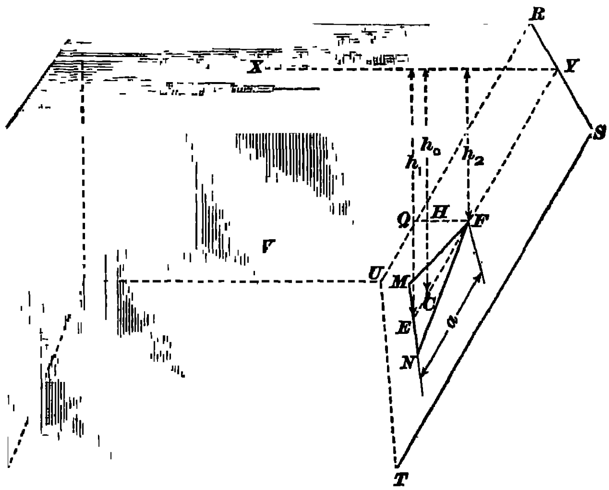


FIG 11

Fig. 11, with its vertex F above its base will be considered The line EFY is perpendicular to the base MN The altitude FE is denoted by a The rest of the notation is similar to that in Fig. 10. The center of pressure is on the line FE, and its depth below the surface of the liquid is given by the formula

$$h_c = \frac{3\,h_1^2}{2\,(2\,h_1 + h_2)} + \frac{h_2}{2} \qquad (1)$$

The distance FC is determined in the same way as for the rectangle treated in Art **26**, the result being

$$FC = \frac{h_c - h_2}{h_1 - h_2} \times a \qquad (2)$$

29. If the vertex is flush with the surface of the liquid, $h_2 = 0$, and the two formulas of the preceding article become

$$h_c = \frac{3h_1^2}{4h_1} = \tfrac{3}{4}h_1 \qquad (1)$$

$$FC = \frac{h_c}{h_1} \times a = \tfrac{3}{4}a \qquad (2)$$

30. Let, now, the vertex be below the base, as shown in Fig 12. With the same notation as before,

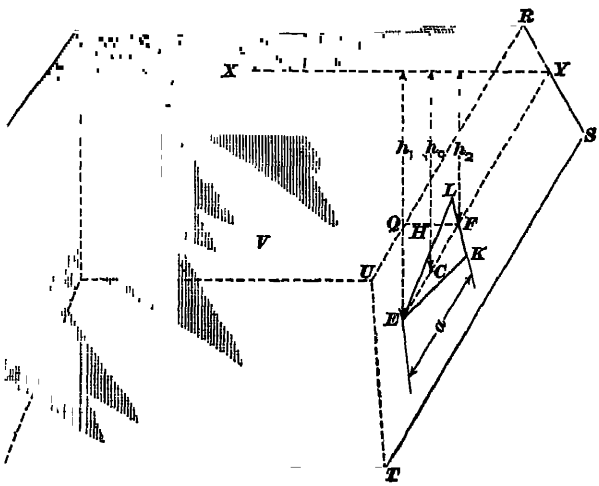


FIG. 12

$$h_c = \frac{3h_2^2}{2(2h_2 + h_1)} + \frac{h_1}{2} \qquad (1)$$

$$FC = \frac{h_c - h_2}{h_1 - h_2} \times a \qquad (2)$$

31. If the base is on the surface of the liquid, $h_2 = 0$, and the two formulas of the preceding article become

$$h_c = \tfrac{1}{2}h_1 \qquad (1)$$

$$FC = \frac{h_c}{h_1} \times a = \tfrac{1}{2}a \qquad (2)$$

32. Center of Pressure of Circular Surface.—Let MN, Fig. 13, be a circular surface. The line EFY contains the lowest point E and the highest point F of the circle.

The diameter of the circle is denoted by d. The rest of the notation is the same as in previous articles.

The center of pressure is on the diameter EF, and the distances h_c and FC are given by the formulas

$$h_c = \tfrac{1}{2}(h_1 + h_2) + \frac{d^2}{8(h_1 + h_2)} \qquad (1)$$

$$FC = \frac{h_c - h_2}{h_1 - h_2} \times d \qquad (2)$$

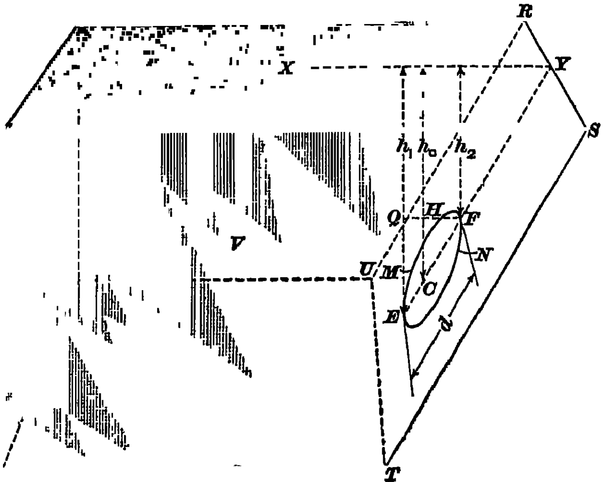


FIG 13

33. If the circle just touches the surface of the liquid, $h_2 = 0$, and formulas **1** and **2** of the preceding article become

$$h_c = \tfrac{1}{2} h_1 + \frac{d^2}{8 h_1} \qquad (1)$$

$$FC = \frac{h_c}{h_1} \times d = \tfrac{1}{2} d + \frac{d^2}{8 h_1^2} = \tfrac{1}{2} d\left[1 + \left(\frac{d}{2 h_1}\right)^2\right] \qquad (2)$$

EXAMPLE.—The vertical circular plate a, Fig. 14, 8 inches in diameter, covers an opening of equal size. The center G is 24 inches below the surface of the water, and 20 inches below the hinge b. What horizontal force applied at c is sufficient to move the plate?

SOLUTION.—Area of plate is 50 266 sq. in Head above center of gravity is 2 ft. Then (Art. **24**), the total pressure on the plate is

$$\frac{50\ 266}{144} \times 62.5 \times 2 = 43.63 \text{ lb.}$$

Here, $h_1 = 24 + \frac{8}{2} = 28$ in., and $h_2 = 24 - \frac{8}{2} = 20$ in. The depth of the center of pressure below the surface is, by formula **1**, Art. **32**, $h_c = 24.167$ in., nearly. The center of pressure is .167 in. below the center of gravity G, and, therefore, 20.167 in. below the hinge b. The moment of P, the total pressure, about the hinge must be equal to the moment of F about the hinge, hence,

$$F \times 40 = P \times 20.167 = 43.63 \times 20.167 = 879.89 \text{ in.-lb.};$$

whence, $F = 879.89 \div 40 = 22$ lb., nearly. Ans.

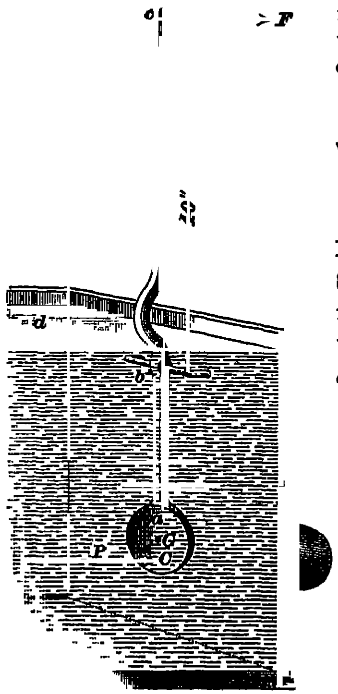


FIG. 14

34. Plates or Gates With Liquid Pressure on Both Sides.—The plate AB, Fig. 15 is subjected to water pressure with the level a' on one side and a'' on the other. The total pres-

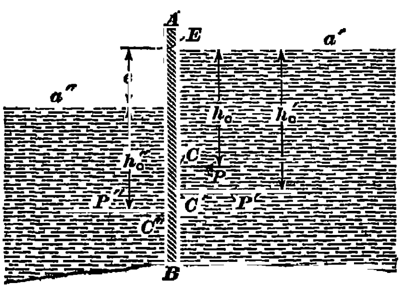


FIG. 15

sure on one side is P', acting through the center of pressure C'; the total pressure on the other side is P'', acting through the center of pressure C''. The heads on the centers of pressures C' and C'' are denoted by h_c' and h_c'', respectively; and the difference between the levels a' and a'' is denoted by e. The two horizontal forces P' and P'' have a resultant P whose magnitude is $P' - P''$. Let h_c be the depth, below a', of the point of application C of this resultant. Taking moments about the point E, we have,

$$P h_c = P' h_c' - P''(h_c'' + e);$$

whence,

$$h_c = \frac{P' h_c' - P''(h_c'' + e)}{P' - P''}$$

EXAMPLE.—Let the gate in Fig. 15 be rectangular and 6 feet wide; let the depth BE be 9 feet and the difference e, 2.4 feet. Required the pressures P', P'', P, and the distance h_c.

SOLUTION.—From formula of Art. **24**, the total pressure on the right side is

$$P' = 9 \times 6 \times 62.5 \times \frac{9}{2} = 15{,}188 \text{ lb. Ans.}$$

The total pressure on the left side is

$$P'' = 6 \times 6.6 \times 62.5 \times \frac{6.6}{2} = 8{,}167.5 \text{ lb. Ans.}$$

The resultant force is

$$P = 15{,}188 - 8{,}167.5 = 7{,}020.5 \text{ lb.}$$

From formula **1** of Art. **27**, $h_c' = \frac{2}{3} \times 9 = 6$, and $h_c'' = \frac{2}{3} \times 6.6 = 4.4$, also, $e = 9 - 6.6 = 2.4$. Substituting in formula of Art. **34**,

$$h_c = \frac{15{,}188 \times 6 - 8{,}167.5 \times (4.4 + 2.4)}{7{,}020.5} = 5.07 \text{ ft. Ans.}$$

35. Assume the plate B, Fig. 16, to be entirely below the water levels a and b, and, therefore, subjected to pressure on both sides. Let the triangle OMN be drawn to represent the variation in pressure due to the weight of the water on the left-hand side (see Art. **20**), and let the triangle $O'M'N'$ be drawn to represent the variation of pressure due to the weight of the water on the right-hand side. Since the angles MON and $M'O'N'$ are equal, the tangent of each being w (Art. **20**), it follows that $O'N'$ and ON are parallel. Take $MO'' = M'O'$, and through O'' draw $O''N''$ parallel to $O'N'$; then, the triangles $O'M'N'$ and $O''MN''$ are equal. Now, the intensity of pressure on the left of the gate, at any level, as x, below the surface, is given by the intercept KE, and that on the right by the intercept $DE = D'E'$. The resultant intensity of pressure at that point is the difference $KE - DE = KD$. Since ON and $O''N''$ are parallel, the horizontal intercepts between them are equal, which means that the intensity of pressure is the

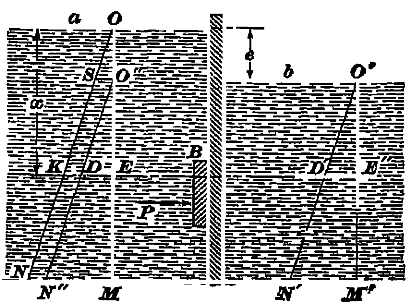


FIG. 16

same at all points of the plate B It follows that the center of pressure for such a submerged plate coincides with its center of gravity.

The total resultant pressure P on the plate is found as follows: The intercept $S O''$ represents the intensity of pressure on the left side at the level b, that is, for the head e, which is the difference between the levels a and b. Hence, since the intensity of pressure on B is everywhere equal to KD, or SO'', or we, it follows that, if the area of B is denoted by A, the total pressure is

$$P = A w e$$

EXAMPLE —In the example of Art 33 (see Fig 14), suppose there is water on the right-hand side of the partition If the level of this water surface is 8 inches lower than that of the level d, find the total resultant pressure on the plate, and also the force F

SOLUTION —The head $e = 8$ in $= \frac{2}{3}$ ft The total pressure is

$$A w e = 50\ 266 \times .434 \times \frac{2}{3} = 14\ 54 \text{ lb} \quad \text{Ans}$$

This pressure acts through the center of gravity, 20 in below the hinge, hence, taking moments about the hinge,

$$F \times 40 = P \times 20,$$

whence, $$F = \tfrac{20}{40} P = \tfrac{1}{2} P = \frac{14\ 54}{2} = 7.27 \text{ lb} \quad \text{Ans.}$$

36. Pressure on Surfaces That Are Not Plane.—If the surface sustaining liquid pressure is not plane, but is

(a) (b) (c)

FIG 17

curved or irregular, *the total pressure on it in any direction, neglecting the pressure due to the weight of the liquid, is the same as the total pressure would be on the projection of the surface on a plane at right angles to the given direction.* To illustrate this statement, consider the three pistons in Fig. 17. At (a) is shown a piston with a curved end, at (b), a piston with an irregular end; and at (c), a piston with a flat end. In each case, the projection of the surface sustaining pressure,

on a plane perpendicular to the piston rod, is the circular cross-section of the cylinder, and if the pressures per unit of area in the cylinders are the same, the pressure on the face of the piston is the same in each case. It is assumed that the pressure per unit of area is the same at all points of the surface; that is, the change of pressure due to the varying depth of the liquid is neglected With this restriction, the law applies to all fluids, both liquid and gaseous.

If the curved or irregular surface forms a part of the wall of a vessel, or is submerged, so that the pressure on it is due to the weight of the liquid, the law of Art. **24** is not true except for a projection on a vertical plane.

37. The whole subject of pressure in any direction on a curved or irregular surface may be summarized as follows:

1. If the external pressure is so great that the pressure due merely to the weight of the liquid may be neglected, the total pressure in any direction is equal to the product of the projection of the surface on a plane perpendicular to that direction, and the pressure per unit of area.

2. When the pressure is due wholly, or in part, to the weight of the liquid, as in the case of submerged surfaces, the total pressure in any direction cannot, in general, be determined except for regularly curved surfaces, such as spheres, cylinders, cones, etc., and for these the computations are difficult and must be made by advanced mathematics.

3. In one direction, however—the horizontal—the total pressure on a surface is easily found. It is precisely the same as the horizontal pressure on the projection of the given surface on a vertical plane.

EXAMPLES FOR PRACTICE

1 What is the pressure on a layer of water 33 feet below the surface? Ans 14.32 lb. per sq. in.

2 What elevation of a reservoir is required to make the pressure in a water main 40 pounds per square inch? Ans. 92.16 ft

3. A vertical triangular plate forms the side of a vessel that contains water The base of the triangle is 4 feet long and lies at the

water level; the vertex is $4\frac{1}{2}$ feet below the water level (*a*) What is the total pressure on the triangle? (*b*) What is the distance of the center of pressure below the water surface? Ans $\begin{cases}(a)\ 843\ 75\text{ lb}\\(b)\ 2\frac{1}{4}\text{ ft}\end{cases}$

4. A circular plate 2 feet in diameter is held vertically so that its upper edge is 5 feet below the liquid surface Find the depth of the center of pressure Ans 6 ft $\frac{1}{2}$ in

5 In the gate shown in Fig 15, the depth of the head-water is 7 feet, and that of the tail-water is 5 feet (*a*) What is the resultant pressure on the gate per foot of length? (*b*) How far above the bottom B is the line of action of the resultant? Ans $\begin{cases}(a)\ 750\text{ lb}\\(b)\ 3\ 03\text{ ft}\end{cases}$

6 The diameter of the plunger of a hydraulic press used in an engineering establishment is 12 inches. Water is forced into the cylinder of the press by means of a small pump having a plunger whose diameter is $\frac{3}{4}$ inch What pressure is exerted by the large plunger when the force acting on the small plunger is 125 pounds? Ans 32,000 lb.

BUOYANT EFFORT OF LIQUIDS

IMMERSION AND FLOTATION

38. Principle of Archimedes.—In a mass of liquid at rest, suppose a part MN, Fig. 18, of the liquid to become solid without changing its form or density. This solid part, having the same density as before, will be held in equilibrium, or remain at rest. Let B denote the weight of the solid part; D, the total downward pressure; and U, the total upward pressure on this part. The ordinary static conditions of equilibrium require that the upward force U should balance the downward forces D and B; that is,

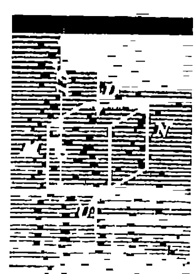


FIG. 18

$$U = D + B;$$

whence,
$$U - D = B \qquad (1)$$

The difference $U - D$ between the upward and the downward pressure is called the **buoyant effort** of the body MN.

Suppose, now, that the imaginary solid part of the water is replaced by an actual solid body having precisely the same

shape and volume, and let the weight of this body equal W. The solid will be subjected to the same vertical pressures D and U as was the part MN of the liquid. Consider, now, the vertical forces acting on it. The weight W and the pressure D are downwards, and tend to cause the body to sink; the pressure U is upwards, and tends to cause the body to rise. The resultant downward force is

$$D + W - U = W - (U - D) = W - B \qquad (2)$$

The weight B is the weight of a mass of liquid whose volume is equal to that of the solid, or, what is the same thing, B is the weight of the liquid displaced by the solid. The following principle may therefore be stated:

When a solid body is immersed in a liquid, a buoyant effort equal to the weight of the liquid displaced acts upwards and opposes the action of gravity. The weight of a body, as shown by a scale, is decreased by an amount equal to the buoyant effort, that is, by an amount equal to the weight of liquid displaced. This principle is called the **principle of Archimedes**, from the name of its discoverer.

EXAMPLE.—In Fig. 18 is shown a cube immersed in water. The edge is 6 inches, the sides are vertical, and the lateral pressures are balanced. On the upper face, there is a downward pressure D due to the head of water of 15 inches, and on the lower face there is an upward pressure U due to the head of 21 inches. To find the buoyant effort.

SOLUTION.—The downward pressure due to the head of 15 in., or $1\frac{1}{4}$ ft., is

$$6^2 \times .434 \times 1\tfrac{1}{4} = 19.53 \text{ lb.}$$

The upward pressure on the lower face due to a head of 21 in., or $1\frac{3}{4}$ ft., is

$$6^2 \times .434 \times 1\tfrac{3}{4} = 27.34 \text{ lb.}$$

The buoyant effort of the body is, therefore,

$$27.34 - 19.53 = 7.81 \text{ lb.} \quad \text{Ans.}$$

Also, the buoyant effort is equal to the weight of the water displaced by the body. The volume of the cube is $(\frac{1}{2})^3$ or $\frac{1}{8}$ cu. ft. The weight of the water displaced is

$$\tfrac{1}{8} \times 62.5 = 7.81 \text{ lb.} \quad \text{Ans.}$$

39. The principle of Archimedes may be experimentally verified with a beam balance, as shown in Fig. 19. From one scale pan, suspend a hollow metallic cylinder t, and

below that a solid cylinder *a* of the same size as the hollow part of the upper cylinder. Put weights in the other scale pan until they exactly balance the two cylinders If *a* is immersed in water, the scale pan containing the weights will descend, showing that *a* has lost some of its weight. Now fill *t* with water; the volume of water that can be poured into *t* is obviously equal to that displaced by *a*. The scale pan containing the weights will rise gradually until *t* is filled, when the scales will balance again

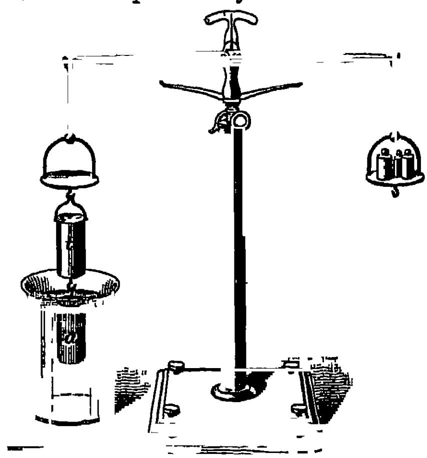


FIG 19

40. If a body immersed in a liquid has the same weight as the liquid it displaces, then $W = B$, the resultant vertical force $W - B$ is zero, and the body will remain at rest at any depth below the surface. If the body is heavier than the liquid it displaces, W is greater than B, and the resultant vertical force $W - B$ is downwards, hence, the body will sink to the bottom. If the body is lighter than the liquid it displaces, B is greater than W, the resultant vertical force $B - W$ is upwards, and the body will rise to the surface.

41. An interesting experiment in confirmation of the conclusions just derived may be performed as follows: Drop an egg into a glass jar filled with fresh water. The mean density of the egg being a little greater than that of fresh water, the egg will fall to the bottom. Now dissolve salt in the water, stirring the mixture; as soon as the salt water becomes denser than the egg the latter begins to rise. If fresh water is then poured in until the egg and water have the same density, the egg will remain stationary in any position it may be placed below the surface of the water.

42. Floating Bodies.—A body lighter than a liquid, bulk for bulk, rises to the surface, when immersed, and floats. For equilibrium, the buoyant effect B must be just equal to the weight W of the body. But, since B is the weight of liquid displaced, it follows that *the weight of the liquid displaced by a floating body is equal to the weight of the body.*

The depth of a floating body in a liquid depends on the relative weights of equal volumes of the body and the liquid. If the body is nearly as heavy as the liquid, it will sink until it displaces nearly its whole volume; if very light, compared with the liquid, the larger part of the body will be above the liquid surface. For example, the density of ice being about nine-tenths that of water, about one-tenth of an iceberg appears above the surface and nine-tenths is submerged. The density of pine is about one-half that of water; hence, about one-half of a floating pine log is submerged, and one-half is above water.

EXAMPLE 1.—Water-tight canvas air bags are used for raising sunken ships These bags are sunk when collapsed, attached to the ship by divers and then filled with air from the pumps above. (*a*) If the capacity of a bag is 200 cubic feet, what is the buoyant effort? (*b*) How many bags will be required to lift 600 tons?

SOLUTION.—(*a*) The weight of the bag and the enclosed air may be neglected. The buoyant effort is the weight of the water displaced by the full bag, that is,

$$200 \times 62.5 = 12{,}500 \text{ lb. Ans.}$$

(*b*) To raise 600 tons,

$$\frac{600 \times 2{,}000}{12{,}500} = 96 \text{ bags are necessary. Ans.}$$

EXAMPLE 2 —A cast-iron cylinder 14 inches long and 8 inches in diameter is closed at the ends, and the metal is ½ inch thick throughout. Will the cylinder float or sink in water?

SOLUTION.—The volume of the entire cylinder is

$$7854 \times 8^2 \times 14 = 703.72 \text{ cu. in.}$$

The hollow portion has a length of 13 in. and a diameter of 7 in.; its volume is, therefore,

$$.7854 \times 7^2 \times 13 = 500.30 \text{ cu. in.}$$

The volume of metal is

$$703.72 - 500.30 = 203.42 \text{ cu. in.}$$

Taking the weight of cast iron as 450 lb per cu. ft., the weight of the cylinder is

$$450 \times \frac{203.42}{1,728} = 52.97 \text{ lb}$$

If immersed, the cylinder displaces 703.72 cu. in of water, which weighs

$$62.5 \times \frac{703.72}{1,728} = 25.45 \text{ lb}$$

The buoyant effort being less than the weight, the cylinder will sink Ans.

SPECIFIC GRAVITY

43. Definition.—The **specific gravity** of a solid or of a liquid substance is the ratio of the weight of any volume of that substance to the weight of an equal volume of water. This ratio varies slightly with temperature, and the values given by physicists are exactly correct only for a temperature of about 39° F., at which water has its maximum density. For practical purposes, however, it is not necessary to take changes of temperature into account. The abbreviation Sp. Gr is often used for specific gravity.

44. Specific Gravity of Solids Not Soluble in Water.—The principle of Archimedes affords a very easy manner of determining the specific gravity of a solid not soluble in water. The body is first weighed in air; it is then attached to a scale pan and weighed in water. *The difference between the two weights will be the weight of an equal volume of water. The ratio of the weight in air to the difference thus found will be the specific gravity.*

Let W be the weight of the body in air and W' the weight in water; then $W - W'$ is the weight of a volume of water equal to the volume of the solid, and

$$\text{Sp. Gr.} = \frac{W}{W - W'}$$

EXAMPLE.—A body weighs in air $36\frac{1}{4}$ ounces and in water 30 ounces. What is its specific gravity?

SOLUTION.—Here $W = 36\frac{1}{4}$, and $W' = 30$. Substituting in the formula,

$$\text{Sp. Gr.} = \frac{36\frac{1}{4}}{36\frac{1}{4} - 30} = \frac{36\frac{1}{4}}{6\frac{1}{4}} = 5.8. \text{ Ans.}$$

45. If the body is lighter than water, a piece of iron or other heavy substance must be attached to it, sufficiently heavy to sink it. Then the two bodies are weighed together both in air and in water, both are weighed separately in air, and the heavier body in water. Subtracting the combined weight of the bodies in water from their combined weight in air, the result will be the weight of a volume of water equal to the volume of the two bodies. The difference between the weight of the heavy body in air and in water gives the weight of a volume of water equal to the volume of the heavy body. Subtracting this last result from the former, the result will be the weight of a volume of water equal to the volume of the light body. The weight of the light body in air divided by the weight of an equal volume of water, as just determined, is the specific gravity of the light body.

Let W = weight of both bodies in air;
W' = weight of both bodies in water;
w = weight of light body in air;
W_1 = weight of heavy body in air;
W_2 = weight of heavy body in water.

Then, the specific gravity of the light body is given by the formula

$$\text{Sp. Gr} = \frac{w}{(W - W') - (W_1 - W_2)}$$

EXAMPLE.—A piece of cork weighs, in air, 4 8 ounces To it is attached a piece of cast iron weighing 36 ounces in air and 31 ounces in water The weight of the iron and cork together, in water, is 15 8 ounces, what is the specific gravity· (*a*) of cork? (*b*) of cast iron?

SOLUTION.—(*a*) Here $w = 4\ 8$, $W = 40\ 8$, $W' = 15\ 8$; $W_1 = 36$, $W_2 = 31$ Substituting in the formula,

$$\text{Specific gravity is } \frac{4\ 8}{(40\ 8 - 15.8) - (36 - 31)} = \frac{4.8}{20} = .24. \quad \text{Ans.}$$

(*b*) To apply formula of Art **44**, $W = 36$, $W' = 31$.

$$\text{Specific gravity of cast iron is } \frac{36}{36 - 31} = 7\ 2. \quad \text{Ans.}$$

46. Specific Gravity of a Liquid.—To determine the specific gravity of a liquid, proceed as follows: Weigh an empty flask; fill it with water, then weigh it and find the

difference between the two results; this will be equal to the weight of the water. Then weigh the flask filled with the liquid, and subtract the weight of the flask; the result is the weight of a volume of the liquid equal to the volume of the water. The weight of the liquid divided by the weight of the water is the specific gravity of the liquid.

Let w = weight of empty flask,
W = weight of flask when filled with the liquid;
W' = weight of flask when filled with water.

Then, $$\text{Sp. Gr.} = \frac{W - w}{W' - w}$$

EXAMPLE.—A flask when empty weighs 8 ounces; when filled with water, 33 ounces, and when filled with alcohol, 28 ounces. What is the specific gravity of the alcohol?

SOLUTION.—Here $W = 28$, $w = 8$, $W' = 33$. Substituting in the formula,

$$\text{Sp. Gr.} = \frac{28 - 8}{33 - 8} = .8 \quad \text{Ans.}$$

47. Nicholson's Hydrometer.—Instruments called **hydrometers** are in general use for determining quickly and accurately the specific gravity of liquids and of some solids. One of the principal forms is Nicholson's hydrometer, which is shown in Fig. 20. It consists of a hollow cylinder, carrying at its lower end a basket d, heavy enough to keep the apparatus upright when placed in water. At the top of the cylinder is a vertical rod, to which is attached a shallow pan a. The cylinder is made of such size and weight that the apparatus is somewhat lighter than water, and a certain weight W must be placed in the pan to sink it to a given point c on the rod. The body whose specific gravity it is desired to find must weigh less than W. It is placed in the pan a, and enough weight w is added to sink the point c to the water level. It is evident that the weight in air of the given body is $W - w$. The body is now removed from the pan a and placed in the basket d, an

FIG. 20

additional weight being added to sink the point c to the water level. Represent the weight now in the pan by W'. The difference $W' - w$ is the weight of a volume of water equal to the volume of the body. Hence,

$$\text{Sp. Gr.} = \frac{W - w}{W' - w}$$

EXAMPLE —The weight necessary to sink a hydrometer to the point c is 16 ounces, the weight necessary when the body is in the pan a is 7 3 ounces; and when the body is in the basket d, 10 ounces. What is the specific gravity of the body?

SOLUTION —Here $W = 16$, $w = 7\ 3$, $W' = 10$ Substituting in the formula,

$$\text{Specific gravity} = \frac{16 - 7\ 3}{10 - 7\ 3} = 3.22 \quad \text{Ans.}$$

48. Volume of an Irregular Solid.—The principle of Archimedes affords a very easy and accurate method of finding the volume of an irregularly shaped body. Let the weight of the body in air be W, and in water W'. The difference $W - W'$ is, according to the principle of Archimedes, the weight of a volume of water equal to the volume of the body. If this volume is denoted by V, and the weight of water per unit of volume is denoted by W_o, the weight of the volume V is $W_o V$, and, therefore,

$$W_o V = W - W';$$

whence,

$$V = \frac{W - W'}{W_o} \qquad (1)$$

If the volume is expressed in cubic feet, W_o is 62.5 pounds, and, therefore,

$$V = \frac{W - W'}{62\ 5} = .016(W - W') \qquad (2)$$

If the volume is expressed in cubic inches, $W_o = .03617$ pound, and, therefore,

$$V = \frac{W - W'}{.03617} = 27.647(W - W') \qquad (3)$$

EXAMPLE —The weight of a body in air is 96 pounds, and in water, 48 8 pounds What is the volume of the body?

SOLUTION —To apply formula 2, we have $W = 96$ and $W' = 48.8$. Substituting in the formula,

$$V = .016(96 - 48.8) = .76 \text{ cu. ft.} \quad \text{Ans.}$$

49. If the specific gravity of a body is known, the cubical contents of the body can be found by dividing its weight by its specific gravity, and then dividing again by either .03617 or 62 5, according as the volume is desired in cubic inches or in cubic feet.

EXAMPLE —A certain body has a specific gravity of 4 38 and weighs 76 pounds What is the volume of the body in cubic inches?

SOLUTION.— $\frac{76}{4\ 38 \times .03617} = 479.7$ cu. in. Ans.

EXAMPLES FOR PRACTICE

1 If a certain quantity of red lead weighs 5 pounds in air and 4 441 pounds in water, what is its specific gravity? Ans. 8.94

2. A piece of iron weighing 1 pound in air and 861 pound in water is attached to a piece of wood weighing 1 pound in air. When both bodies are placed in water they weigh 2 pound. What is the specific gravity: (a) of the iron? (b) of the wood? Ans. $\begin{cases} (a) & 7\ 194 \\ (b) & 602 \end{cases}$

3. An empty flask weighs 13 ounces; when filled with water, it weighs 22 ounces, and when filled with sulphuric acid, 29.56 ounces. What is the specific gravity of the acid? Ans. 1 84

4. How many cubic feet of brick having a specific gravity of 1.9 are required to make a total weight of 260 pounds? Ans. 2.19 cu. ft.

PNEUMATICS

PROPERTIES OF AIR AND OTHER GASES

1. Pneumatics is that branch of science that treats of the mechanical properties of gases.

2. The distinguishing property of a gas is that, *no matter how small the quantity may be, the gas will always fill the vessel or vessels that contain it.* If a bladder is partly filled with air and placed under a glass jar (called a **receiver**) from which the air has been exhausted, the bladder will immediately expand, as shown in Fig. 1. The force that a gas exerts when confined in a limited space is called **tension.** In this case, the word tension means pressure, and is only used in this sense when referring to gases.

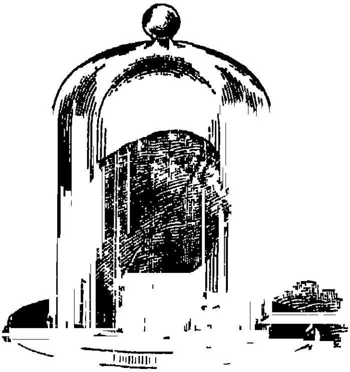


FIG. 1

3. As water is the most common type of fluids, so air is the most common type of gases. It was supposed by the ancients that air was "imponderable," by which was meant that it weighed nothing, and it was not until about the year 1650 that it was proved that air really has weight. A cubic inch of air, under ordinary conditions, weighs about .31 grain. The ratio of the weight of air to that of water is about 1 . 774; that is, air is only $\frac{1}{774}$ as heavy as water.

It was shown in *Hydrostatics* that if a body is immersed in water, and weighs less than the volume of water it displaces, the body will rise and project partly out of the water. The same principle, which is the principle of Archimedes, applies to gases. If a vessel made of light material is filled with a gas lighter than air, so that the total weight of the vessel and gas is less than the weight of the volume of air they displace, the vessel will rise. It is on this principle that balloons are made.

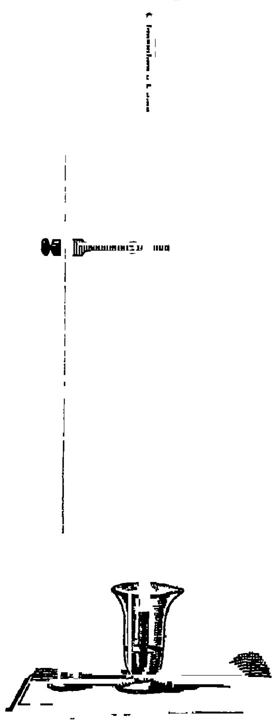


FIG 2

4. Since air has weight, it is evident that the enormous quantity of air that constitutes the atmosphere must exert a considerable pressure on the earth. This is easily proved by taking a long glass tube, closed at one end, and filling it with mercury. If the finger is placed over the open end, so as to keep the mercury from running out, and the tube is inverted and placed in a glass partly filled with the same liquid, as shown in Fig. 2, the mercury in the tube will fall, then rise, and after a few oscillations will come to rest at a height above the top of the mercury in the glass equal to about 30 inches. This height will always be practically the same under the same atmospheric conditions. Now, since the atmosphere has weight, it presses on the upper surface of the mercury in the glass with equal force on every square unit, except on that part of the surface occupied by the tube. According to Pascal's law (see *Hydrostatics*), this pressure is transmitted in all directions. There being nothing in the tube, except the mercury, to counterbalance the upward pressure of the air, the mercury falls in the tube until it exerts an upward pressure on the upper surface of

the mercury in the glass sufficiently great to counterbalance the downward pressure produced by the atmosphere. In order that there may be equilibrium, the pressure of the air per unit of area on the upper surface of the mercury in the glass must equal the pressure (weight) exerted per unit of area by the mercury inside of the tube. Suppose that the area of the inside of the tube is 1 square inch, then, since mercury is 13 6 times as heavy as water, and 1 cubic inch of water weighs about 03617 pound, the weight of the mercurial column is $.03617 \times 13.6 \times 30 = 14.7574$ pounds The actual height of the mercury is a little less than 30 inches, and the actual weight of a cubic inch of distilled water is a little less than .03617 pound. When these considerations are taken into account, the average weight of the mercurial column at the level of the sea is 14.696 pounds, or, as it is usually expressed, 14.7 pounds. Since this weight, exerted on 1 square inch of the liquid in the glass, just produces equilibrium, it is plain that the pressure of the outside air is 14 7 pounds on every square inch of surface. This pressure is often referred to as *one atmosphere*. A pressure of two atmospheres is a pressure of $2 \times 14.7 = 29.4$ pounds per square inch, a pressure of three atmospheres is a pressure of $3 \times 14.7 = 44.1$ pounds per square inch; etc.

5. Vacuum.—Referring to Fig. 2, the space between the upper end of the tube and the upper surface of the mercury in the tube is called a **vacuum**, or empty space. If this space contained a gas of some kind, no matter how small the quantity might be, the gas would expand and fill the space, and its tension would, according to the amount present, cause the column of mercury to fall and become shorter; the space would then be called a **partial vacuum**. If the mercury fell 1 inch, so that the column was only 29 inches high, this would be expressed by saying that there were *29 inches of vacuum;* a fall of 8 inches would be referred to as 22 inches of vacuum; a fall of 16 inches, as 14 inches of vacuum, etc. Hence, when the vacuum gauge of a condensing engine shows 26 inches of vacuum, there is enough air in the

condenser to produce a pressure of $\frac{30-26}{30} \times 14.7 = \frac{4}{30} \times 14.7$ $= 1.96$ pounds per square inch. In all cases where the mercurial column is used to measure a vacuum, the height of the column, in inches, gives the number of inches of vacuum Thus, if the column were 5 inches high, or the vacuum gauge showed 5 inches, the vacuum would be 5 inches

If the tube had been filled with water instead of mercury, the height of the column of water to balance the pressure of the atmosphere would have been about $30 \times 13.6 = 408$ inches = 34 feet This means that if a tube is filled with water, and is inverted and placed in a dish of water in a manner similar to that shown in Fig 2, the resulting height of the column of water will be about 34 feet

FIG 3

6. The **barometer** is an instrument used for measuring the pressure of the atmosphere There are two kinds in general use—the mercurial barometer and the aneroid barometer. The latter was described in *Leveling* The **mercurial barometer** is shown in Fig 3. The principle is the same as in the case of the inverted tube shown in Fig 2 The tube and cup at the bottom are protected by a brass or an iron casing At the top of the tube is a graduated scale that can be read to $\frac{1}{1000}$ inch by means of a vernier. Attached to the casing is an accurate thermometer for determining the temperature of the outside air at the time the barometric observation is taken. This is necessary, since mercury expands when the temperature rises, and contracts when the temperature falls; for this reason a standard temperature is assumed, and all barometer readings are reduced to this temperature. This standard temperature is usually taken as 32° F., at which temperature the height of the mercurial

column is 30 inches. Another correction is made for the altitude of the place above sea level, and a third correction for the effects of capillary attraction. It is not necessary here to go into details regarding these corrections.

7. The pressure of the atmosphere varies with the altitude above sea level, being greater in low than in high places. At the level of the sea, the height of the mercurial column is about 30 inches, at 5,000 feet above the sea, it is 24.7 inches; at 10,000 feet above the sea, it is 20 5 inches; at 15,000 feet above the sea, it is 16.9 inches; at 3 miles, it is 16 4 inches; and at 6 miles above the sea level, it is 8 9 inches.

The density also varies with the altitude; that is, a cubic foot of air at an elevation of 5,000 feet above the sea level does not weigh as much as a cubic foot at sea level. This is proved conclusively by the fact that at a height of $3\frac{1}{2}$ miles the mercurial column measures but 15 inches, indicating that half the weight of the entire atmosphere is below that height It is known that the height of the earth's atmosphere is at least 50 miles; hence, the air just before reaching the limit must be in an exceedingly rarefied state. It is by means of barometers that great heights are measured. The aneroid barometer has the heights marked on the dial, so that it can be read directly. With the mercurial barometer, the heights must be calculated from the reading. (See *Leveling*.)

8. The atmospheric pressure is everywhere present, and presses all objects in all directions with equal force. If a book is laid on the table, the air presses on it in every direction with an equal average force of about 14.7 pounds per square inch. It would seem as though it would take considerable force to raise a book from the table, since, if the size of the book were 8 inches by 5 inches, the pressure on it would be $8 \times 5 \times 14.7 = 588$ pounds But there is an equal pressure beneath the book to counteract the pressure on the top. It would now seem as though it would require a great force to open the book, since there are two pressures

of 588 pounds each, acting in opposite directions, and tending to crush the book, and so it would but for the fact that there is a layer of air between each two leaves acting upwards and downwards with a pressure of 14.7 pounds per square inch. If two metal plates are made as perfectly smooth and flat as it is possible to make them, and the edge of one is laid on the edge of the other, so that one may be slid on the other, and the air thus excluded, it will take an immense force, compared with the weight of the plates, to separate them. This is because the full pressure of 14.7 pounds per square inch is then exerted on each plate with no counteracting equal pressure between the plates

If a piece of flat glass is laid on a flat surface that has been previously moistened with water, it will require considerable force to lift it off the surface. This is due to the fact that the water helps to fill up the pores in the flat surface and glass, and thus creates a partial vacuum between the glass and the surface, thereby reducing the counterpressure beneath the glass.

9. Tension of Gases.—In Art. **5**, it was said that the space above the column of mercury in Fig. 2 was a vacuum, and that if any gas or air were present it would expand, its tension forcing the column of mercury downwards. If enough gas is admitted to cause the mercury to stand at 15 inches, the tension of the gas is evidently $14.7 - 2 = 7.35$ pounds per square inch, since the pressure of the outside air of 14.7 pounds per square inch only balances 15 inches, instead of 30 inches, of mercury, that is, it balances only half as much as it would if there were no gas in the tube; hence, the pressure (tension) of the gas in the tube is 7.35 pounds. If more gas is admitted until the top of the mercurial column is just level with the mercury in the cup, the gas in the tube has then a tension equal to the outside pressure of the atmosphere. Suppose that the bottom of the tube is fitted with a piston, and that the total length of the inside of the tube is 36 inches. If the piston is shoved upwards so that the space occupied by the gas is 18 inches long, instead

of 36 inches, the temperature remaining the same as before, it will be found that the tension of the gas within the tube is 29.4 pounds per square inch. It will be noticed that the volume occupied by the gas is only half that in the tube before the piston was moved, while the pressure is twice as great, since $14.7 \times 2 = 29.4$ pounds. If the piston is shoved up so that the space occupied by the gas is only 9 inches, instead of 18 inches, the temperature still remaining the same, the pressure will be found to be 58.8 pounds per square inch. The volume has again been reduced one-half, and the pressure increased two times, since $29.4 \times 2 = 58.8$ pounds. The space now occupied by the gas is 9 inches long, whereas, before the piston was moved, it was 36 inches long, as the tube was assumed to be of uniform diameter throughout its length, the volume is now $\frac{9}{36} = \frac{1}{4}$ of its original volume, and its pressure is $\frac{58.8}{14.7} = 4$ times its original pressure. Moreover, if the temperature of the confined gas remains the same, the pressure and volume will always vary in a similar way. The law that states these effects is called *Mariotte's law*, and is as follows:

10. Mariotte's Law.—*The temperature remaining the same, the volume of a given quantity of gas varies inversely as the pressure.*

The meaning of this is: If the volume of the gas is diminished to one-half, one-third, one-fifth, etc. of its former volume, the tension will be increased two, three, five, etc. times; or, if the outside pressure is increased two, three, five, etc. times, the volume of the gas will be diminished to one-half, one-third, one-fifth, etc. of its original volume, the temperature remaining constant.

Suppose 3 cubic feet of air to be under a pressure of 60 pounds per square inch in a cylinder fitted with a movable piston; then the product of the volume and pressure is $3 \times 60 = 180$. Let the volume be increased to 6 cubic feet; then the pressure will be 30 pounds per square inch, and $30 \times 6 = 180$, as before. Let the volume be increased to

24 cubic feet; it is then 24 − 3 = 8 times the original volume, and the pressure is one-eighth of the original pressure, or $60 \times \frac{1}{8} = 7\frac{1}{2}$ pounds, and $24 \times 7\frac{1}{2} = 180$, as in the two preceding cases. It will now be noticed that, if a gas is allowed to expand without change of temperature, *the product of any pressure and the corresponding volume is the same as for any other pressure and the corresponding volume* If the air were compressed, the same result would be obtained.

Let p = pressure corresponding to volume v;

p_1 = pressure corresponding to volume v_1.

Then,
$$p\,v = p_1\,v_1$$

Knowing the volume and the pressure for any position of the piston, and the volume for any other position, the pressure may be calculated; or, if the pressure is known for any other position, the volume may be calculated.

EXAMPLE 1 —If 1 875 cubic feet of air is under a pressure of 72 pounds per square inch, what will be the pressure when the volume is increased (*a*) to 2 cubic feet? (*b*) to 3 cubic feet? (*c*) to 9 cubic feet?

SOLUTION —Solving the last equation for p_1, the unknown pressure gives

(*a*) $p_1 = \frac{p\,v}{v_1} = \frac{72 \times 1\,875}{2} = 67\frac{1}{2}$ lb per sq in Ans.

(*b*) $p_1 = \frac{72 \times 1\,875}{3} = 45$ lb per sq in Ans.

(*c*) $p_1 = \frac{72 \times 1\,875}{9} = 15$ lb. per sq. in Ans

EXAMPLE 2 —Ten cubic feet of air has a tension of 5.6 pounds per square inch, what is the volume when the tension is (*a*) 4 pounds? (*b*) 8 pounds? (*c*) 25 pounds? (*d*) 100 pounds?

SOLUTION —Solving the same equation for v_1 gives

(*a*) $v_1 = \frac{p\,v}{p_1} = \frac{5\,6 \times 10}{4} = 14$ cu ft Ans

(*b*) $v_1 = \frac{5\,6 \times 10}{8} = 7$ cu ft Ans

(*c*) $v_1 = \frac{5\,6 \times 10}{25} = 2\,24$ cu ft Ans

(*d*) $v_1 = \frac{5\,6 \times 10}{100} = 56$ cu ft Ans.

11. For the same quantity of gas, the weight per unit of volume varies inversely as the volume. For example, if

1 pound of gas occupies a volume of 4 cubic feet, its weight per cubic foot will be $\frac{1}{4}$ pound. If it occupies a volume twice as large, or 8 cubic feet, its weight per cubic foot will be $\frac{1}{8}$ pound, or only one-half of what it was before. In general, if, the temperature of a fixed quantity of gas remaining the same, the weights per unit of volume when the gas occupies the volumes v and v_1, respectively, are denoted by w and w_1, then,

$$\frac{w}{w_1} = \frac{v_1}{v}$$

and

$$w\,v = w_1 v_1 \qquad (1)$$

Also, since

$$\frac{v_1}{v} = \frac{p}{p_1}$$

$$\frac{w}{w_1} = \frac{p}{p_1}$$

and

$$w\,p_1 = w_1 p \qquad (2)$$

EXAMPLE 1 —The weight of 1 cubic foot of air at a temperature of 60° F , and under a pressure of 1 atmosphere (14.7 pounds per square inch), is 0763 pound, what would be the weight per cubic foot if the volume were compressed until the tension was 5 atmospheres, the temperature still being 60° F ?

SOLUTION.—Applying formula **2**, $1 \times w_1 = 5 \times .0763$. Hence, $w_1 = .3815$ lb. per cu. ft. Ans.

EXAMPLE 2.—If in the last example the air had expanded until the tension was 5 pounds per square inch, what would have been its weight per cubic foot?

SOLUTION.—Here, $p = 14\ 7$, $p_1 = 5$, and $w = 0763$ Hence, applying formula **2**, $14.7 \times w_1 = 5 \times .0763$, whence $w_1 = \frac{.3815}{14.7} = 02595$ lb. per cu. ft. Ans.

EXAMPLE 3 —If 6.75 cubic feet of air, at a temperature of 60° F. and a pressure of 1 atmosphere, is compressed to 2 25 cubic feet (the temperature still remaining 60° F.), what is the weight of a cubic foot of the compressed air?

SOLUTION —Applying formula **1**, $6.75 \times .0763 = 2\ 25 \times w_1$. Hence,

$$w_1 = \frac{6.75 \times .0763}{2\ 25} = 2289 \text{ lb. per cu. ft. Ans.}$$

12. Manometers and Gauges.—There are two ways of measuring the pressure of a gas: by means of an instrument called a **manometer**, and by means of a **gauge.**

The manometer generally used is practically the same as a mercurial barometer, except that the tube is much longer, so that pressures equal to several atmospheres may be measured, and is enlarged and bent into a **U** shape at the lower end, both the lower and the upper ends are open, the lower end being connected to the vessel containing the gas whose pressure it is desired to measure. The gauge is so common that no description of it will be given here. With both the manometer and the gauge, the pressure recorded is the amount by which the pressure being measured exceeds the atmospheric pressure, and is called **gauge pressure.** To find the total pressure, or **absolute pressure,** the atmospheric must be added to the gauge pressure. In all formulas in which the pressure of a gas or steam occurs, the absolute pressure must be used, unless the gauge pressure is distinctly specified as the proper one to use. For convenience, all pressures given in this Section and in the questions referring to it will be absolute pressures, and the word "absolute" will be omitted, to avoid repetition.

13. In all that has been said, it has been stated that the temperature was constant; the reason for this will now be explained. Suppose 5 cubic feet of air to be confined in a cylinder placed in a vacuum, so that there will be no pressure due to the atmosphere, and suppose the cylinder to be fitted with a piston weighing, say, 100 pounds, and having an area of 10 square inches. The tension of the air will be $\frac{100}{10} = 10$ pounds per square inch. Suppose, now, that the air, originally at a temperature of 32° F., is heated until its temperature is 33° F.—that is, until its temperature is raised 1°. It will be found that the piston has risen a certain amount, and, consequently, the volume has increased, while the pressure remains the same as before, or 10 pounds per square inch. If more heat is applied, until the temperature of the gas is 34° F., it will be found that the piston has again risen, and the volume again increased, while the pressure still remains the same. It will be found that for every increase in temperature there will be a corresponding increase

of volume. The law that expresses this change is called *Gay-Lussac's law*, and is as follows:

14. Gay-Lussac's Law.—*If the pressure remains constant, every increase in temperature of 1° F. produces in a given quantity of gas an expansion of $\frac{1}{492}$ of its volume at 32° F.*

If the pressure remains constant, it will also be found that every decrease of temperature of 1° F. will cause a decrease of $\frac{1}{492}$ of the volume at 32° F.

Let v_0 = volume of any quantity of gas at 32° F.;
v = volume of gas at temperature t;
v_1 = volume of gas at temperature t_1.

It is assumed that the pressure of the gas remains unchanged. Then, in passing from the temperature 32° to the temperature t, the volume of the gas will be increased algebraically by the amount $\frac{1}{492} v_0 (t - 32) = \frac{t - 32}{492} v_0$.

Therefore,

$$v = v_0 + \frac{t - 32}{492} v_0 = \frac{492 v_0 + (t - 32) v_0}{492}$$

or, reducing, $$v = v_0 \frac{460 + t}{492} \qquad (1)$$

Likewise, $$v_1 = v_0 \frac{460 + t_1}{492} \qquad (a)$$

Dividing formula **1** by equation (a),

$$\frac{v}{v_1} = \frac{460 + t}{460 + t_1}$$

whence $$v = v_1 \frac{460 + t}{460 + t_1} \qquad (2)$$

EXAMPLE.—If 5 cubic feet of air at a temperature of 45° is heated under constant pressure up to 177°, what is the final volume of the air?

SOLUTION.—To apply formula **2**, we have $v_1 = 5$; $t_1 = 45$; $t = 177$. Therefore,

$$v = 5 \times \frac{460 + 177}{460 + 45} = 6.307 \text{ cu. ft} \quad \text{Ans.}$$

15. Suppose that a certain volume of gas is confined in a vessel so that it cannot expand; in other words, suppose that the piston of the cylinder before mentioned is fastened so that it cannot move. Let a gauge be placed on

the cylinder so that the tension of the confined gas can be registered If the gas is heated, it will be found that, for every increase of temperature of 1° F , there will be a corresponding increase of $\frac{1}{492}$ of the tension That is, the volume remaining constant, the tension increases $\frac{1}{492}$ of the original tension for every degree rise of temperature.

Let p = tension of gas at temperature t,

p_1 = tension of gas at temperature t_1.

Then, as in the preceding article,

$$p = p_1 \frac{460 + t}{460 + t_1}$$

EXAMPLE —If a certain quantity of air is heated under constant volume from 45° to 177°, what is the resulting tension, the original tension being 14 7 pounds per square inch?

SOLUTION —Applying the formula,

$$p = 14\ 7 \times \frac{460 + 177}{460 + 45} = 18\ 542 \text{ lb per sq in} \quad \text{Ans}$$

16. According to the modern and now generally accepted theory of heat, the atoms and molecules of all bodies are in an incessant state of vibration. The vibratory movement in liquids is faster than in solids, and in gases faster than in either of the other two Any increase of heat increases the vibrations, and a decrease of heat decreases them From experiments and advanced mathematical investigations, it has been concluded that at 460° below zero, on the Fahrenheit scale, all these vibrations cease. This point is called the **absolute zero,** and all temperatures reckoned from this point are called **absolute temperatures.** The point of absolute zero has never been reached, nevertheless, it has a meaning, and is used in many formulas, absolute temperatures being usually denoted by T. Ordinary temperatures are denoted by t. When the word temperature alone is used, it refers to the ordinary way of measuring temperatures, but when absolute temperature is specified 460° F. must be added to the ordinary temperature The absolute temperature corresponding to 212° F. is 460 + 212 = 672° F. If the absolute temperature is given, the ordinary temperature may be found by subtracting 460 from the absolute temperature.

Thus, if the absolute temperature is 520° F., the ordinary temperature is 520° − 460° = 60°.

17. Let p = pressure, in pounds per square inch, of W pounds of air;
V = volume of air, in cubic feet,
T = absolute temperature of air.

It is shown in advanced works on the theory of heat that these quantities are related by the following equation:

$$p\,V = .37\ W\,T$$

EXAMPLE 1.—The pressure on 9 cubic feet of air weighing 1 pound is 20 pounds per square inch, what is the ordinary temperature of the air?

SOLUTION.—Here, $W = 1$. Substituting the other values in the formula gives $20 \times 9 = .37\ T$, hence, $T = \frac{180}{.37} = 486.5°$, nearly. $486.5° - 460° = 26.5°$, the ordinary temperature. Ans.

EXAMPLE 2.—What is the volume of 1 pound of air whose temperature is 60° F under a pressure of 1 atmosphere?

SOLUTION.—Here, $W = 1$, and $T = 460 + 60 = 520$, therefore, $14.7 \times V = .37 \times 520$, whence $V = \frac{.37 \times 520}{14.7} = 13.088$ cu. ft. Ans.

EXAMPLE 3.—If 3 cubic feet of air weighing .35 pound is under a pressure of 48 pounds per square inch, what is the ordinary temperature of the air?

SOLUTION.—Applying the formula, $48 \times 3 = .37 \times .35 \times T$, whence, $T = \frac{48 \times 3}{.37 \times .35} = 1{,}112.0°$. Then, $1{,}112.0° - 460° = 652°$. Ans.

EXAMPLE 4.—What is the weight of 1 cubic foot of air at a temperature of 32°, and under a pressure of 1 atmosphere?

SOLUTION.—Here $T = 460° + 32° = 492°$; $V = 1$; and $p = 14.7$ (Art. 4). Substituting these values in the formula gives $14.7 \times 1 = .37 \times 492 \times W$, whence,

$$W = \frac{14.7}{.37 \times 492} = .080751 \text{ lb.} \quad \text{Ans.}$$

If the pressure is taken as 14.696 lb. per sq. in., the weight of 1 cu. ft. of air, at 32° and atmospheric pressure, is found to be

$$\frac{14.696}{.37 \times 492} = .08073 \text{ lb., nearly}$$

EXAMPLES FOR PRACTICE

1 A vessel contains 25 cubic feet of gas at a pressure of 18 pounds per square inch, if 125 cubic feet of gas having the same pressure is forced into the vessel, what will be the resulting pressure?

Ans 108 lb. per sq. in.

2 A pound of air has a temperature of 126°, and a pressure of 1 atmosphere, what volume does it occupy? Ans. 14 75 cu. ft.

3 A certain quantity of air has a volume of 26 7 cubic feet, a pressure of 19.3 pounds per square inch, and a temperature of 42°; what is its weight? Ans. 2 77 lb

4 A receiver contains 180 cubic feet of gas at a pressure of 20 pounds per square inch, if a vessel holding 12 cubic feet is filled from the receiver until its pressure is 20 pounds per square inch, what will be the pressure in the receiver? Ans $18\frac{2}{3}$ lb per sq in.

5 Ten cubic feet of air, having a pressure of 22 pounds per square inch and a temperature of 75°, is heated until the temperature is 300°, the volume remaining the same, what is the new pressure?

Ans 31 25 lb per sq in.

THE MIXING OF GASES

18. If two liquids that do not act chemically on each other are mixed together and allowed to stand, it will be found that after a time the two liquids have separated, and that the heavier has fallen to the bottom. If two equal vessels containing gases of different densities are put in communication with each other, it will be found that after a short time the gases have become mixed in equal proportions. If one vessel is higher than the other, and the heavier gas is in the lower vessel, the same result will occur. The greater the difference of the densities of the two gases, the faster they will mix. It is assumed that no chemical action takes place between the two gases. When the two gases have the same temperature and pressure, the pressure of the mixture will be the same. This is evident, since the total volume has not been changed, and, unless the volume or temperature changes, the pressure cannot change This property of the mixing of gases is a very valuable one, since, if gases acted like liquids, carbonic-acid gas (the result

of combustion), which is $1\frac{1}{2}$ times as heavy as air, would remain next to the earth, instead of dispersing into the atmosphere, the result being that no animal life could exist.

19. Mixture of Equal Volumes of Gases Having Unequal Pressures.—*If two gases having equal volumes and temperatures, but different pressures, are mixed in a vessel whose volume equals one of the equal volumes of the gases, the pressure of the mixture will be equal to the sum of the two pressures, provided that the temperature remains the same as before.*

20. Mixture of Two Gases Having Unequal Volumes and Pressures.—Let v and p be the volume and pressure, respectively, of one of the gases.

Let v_1 and p_1 be the volume and pressure, respectively, of the other gas.

Let V and P be the volume and pressure, respectively, of the mixture. Then, if the temperature remains the same,

$$VP = vp + v_1 p_1$$

That is, *if the temperature is constant, the volume after mixture, multiplied by the resulting pressure, equals the volume of one gas before mixture multiplied by its pressure, plus the volume of the other gas multiplied by its pressure.*

EXAMPLE —Two gases at the same temperature, having volumes of 7 cubic feet and $4\frac{1}{2}$ cubic feet, and pressures of 27 pounds and 18 pounds per square inch, respectively, are mixed together in a vessel whose volume is 10 cubic feet. What is the resulting pressure?

SOLUTION.—Applying the preceding formula, $PV = pv + p_1 v_1$, or $P \times 10 = 27 \times 7 + 4\frac{1}{2} \times 18$. Hence, $P = \frac{189 + 81}{10} = 27$ lb. per sq. in. Ans.

21. Mixture of Two Volumes of Air Having Unequal Pressures, Volumes, and Temperatures.—If a body of air having a temperature t_1, a pressure p_1, and a volume v_1 is mixed with a volume of air having a temperature t_2, a pressure p_2, and a volume v_2, to form a volume V having a pressure P and a temperature t, then, either the new temperature t, the new volume V, or the new pressure P may be found, if the other two quantities are known, by the following

formula, in which T_1, T_2, and T are the absolute temperatures corresponding to t_1, t_2, and t:

$$PV = \left(\frac{p_1 v_1}{T_1} + \frac{p_2 v_2}{T_2}\right) T$$

EXAMPLE —Five cubic feet of air having a tension of 30 pounds per square inch and a temperature of 80° F are compressed, together with 11 cubic feet of air having a tension of 21 pounds per square inch and a temperature of 45° F, in a vessel whose cubical contents are 8 cubic feet The new pressure is required to be 45 pounds per square inch What must be the temperature of the mixture?

SOLUTION —Substituting in the formula, $45 \times 8 = \left(\frac{30 \times 5}{540} + \frac{21 \times 11}{505}\right) \times T$, or $360 = .7352\ T$ Hence, $T = \frac{360}{.7352} = 489.66°$, nearly, and $t = 29.66°$ Ans

EXAMPLES FOR PRACTICE

1 Two vessels contain air at pressures of 60 and 83 pounds per square inch The volume of each vessel is 8.47 cubic feet If all the air in both vessels is removed to another vessel, and the new pressure is 100 pounds per square inch, what is the volume of the vessel, the temperature remaining unchanged? Ans. 12.11 cu ft

2 A vessel contains 11.83 cubic feet of air at a pressure of 33.3 pounds per square inch It is desired to increase the pressure to 40 pounds per square inch by supplying air from a second vessel that contains 19.6 cubic feet of air at a pressure of 60 pounds per square inch What will be the pressure in the second vessel after the pressure in the first has been raised to 40 pounds per square inch? Ans. 55.96 lb. per sq in.

3 If 4.8 cubic feet of air having a tension of 52 pounds per square inch and a temperature of 170° is mixed with 13 cubic feet having a tension of 78 pounds per square inch and a temperature of 265°, what must be the volume of the vessel containing the mixture in order that the tension of the mixture may be 30 pounds per square inch and the temperature 80°? Ans. 32.31 cu ft.

PNEUMATIC MACHINES

THE AIR PUMP

22. The **air pump** is an instrument for removing air from an enclosed space. A section of the principal parts is shown in Fig. 4, and a view of the complete instrument is given in Fig. 5. The closed vessel R is called the **receiver,**

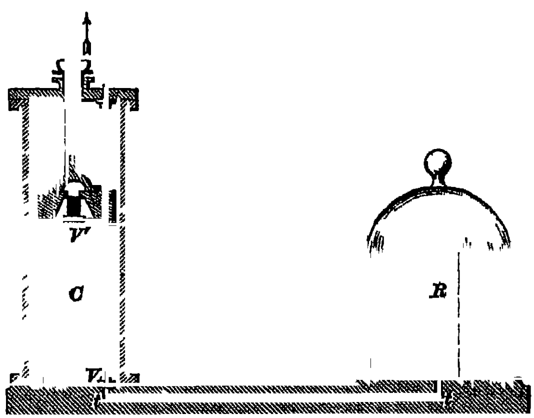


FIG 4

and the space it encloses is that from which it is desired to remove the air. The receiver is usually made of glass, and the edges are ground so as to be perfectly air-tight. When made in the form shown, it is called a **bell-jar receiver.** The receiver rests on a horizontal plate, in the center of which is an opening communicating with the pump cylinder C by means of a bent tube t. The pump piston fits the cylinder accurately, and has a valve V' opening upwards. At the junction of the tube with the cylinder is another valve V, also opening upwards. When the piston is raised,

the valve V' closes, and, since no air can get into the cylinder from above, the piston leaves a vacuum behind it. The pressure on top of V being now removed, the tension of the air in the receiver R causes V to rise, the air in the receiver then expands and occupies the space left empty by the piston, as well as the space in the tube t and the receiver R. The piston is now pushed down, the valve V closes, the valve V' opens, and the air in C escapes. The lower valve V is sometimes supported, as shown in Fig. 4, by a metal rod passing through the piston and fitting it somewhat tightly. When the piston is raised or lowered, this rod moves with it. A button near the upper end of the rod confines its motion to within very narrow limits, the piston sliding on the rod during the greater part of the journey.

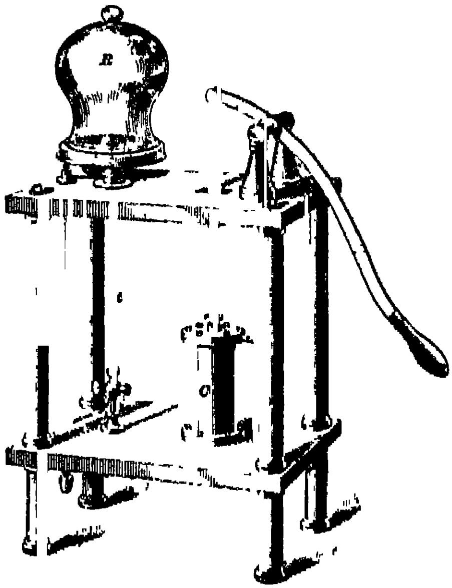
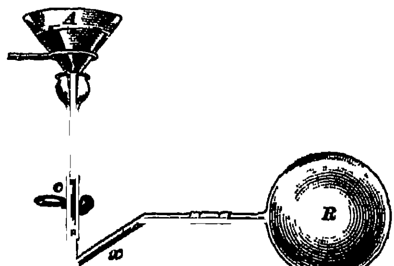


FIG. 5

23. Degrees and Limits of Exhaustion.—Suppose that the volume of R and t together is four times that of C, and that there are, say, 200 grains of air in R and t, and 50 grains in C, when the piston is at the top of the cylinder. At the end of the first stroke, when the piston is again at the top, 50 grains of air in the cylinder C will have been removed, and the 200 grains in R and t will occupy the spaces R, t, and C. The ratio between the sum of the spaces R and t and the total space $R + t + C$ is $\frac{4}{5}$; hence, $200 \times \frac{4}{5} = 160$ grains = the weight of air in R and t after

the first stroke. After the second stroke, the weight of the air in R and t will be $200 \times \frac{4}{5} \times \frac{4}{5} = 200 \times \left(\frac{4}{5}\right)^2 = 200 \times \frac{16}{25} = 128$ grains. At the end of the third stroke, the weight will be $\left[200 \times \left(\frac{4}{5}\right)^2\right] \times \frac{4}{5} = 200 \times \left(\frac{4}{5}\right)^3 = 200 \times \frac{64}{125} = 102.4$ grains. At the end of n strokes, the weight will be $200 \times \left(\frac{4}{5}\right)^n$ It is evident that *it is impossible by this method to remove all the air contained in R and t.* It requires an exceedingly good air pump to reduce the tension of the air in R to $\frac{1}{50}$ inch of mercury. When the air has become so rarefied as this, the valve V' will not lift, and, consequently, no more air can be exhausted



24. Sprengel's Air Pump.—In Fig 6, $c\,d$ is a glass tube longer than 30 inches, open at both ends, and connected by means of India-rubber tubing with a funnel A filled with mercury and supported by a stand. Mercury is allowed to fall into this tube at a rate regulated by a clamp at c. The lower end of the tube $c\,d$ fits in the flask B, which has a spout at the side a little higher than the lower end of $c\,d$; the upper part has a branch at x to which a receiver R can be tightly fixed. When the clamp at c is opened, the first portions of the mercury that run out close the tube and prevent air from entering

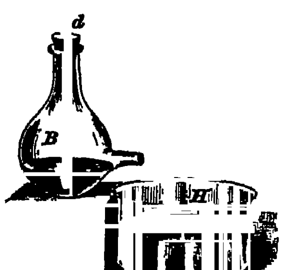


FIG. 6

from below. These drops of mercury act like little pistons, carrying the air in front of them and forcing it out through the bottom of the tube. The air in R expands to fill the tube every time that a drop of mercury falls, thus creating a partial vacuum in R, which becomes more nearly complete as the process goes on. The escaping mercury falls into the dish H, from which it can be poured back into the funnel from time to time. As the exhaustion from R goes on, the mercury rises in the tube $c\,d$ until, when the exhaustion is complete, it forms a continuous column 30 inches high, in other words, it is a barometer whose vacuum is the receiver R. This instrument necessarily requires a great deal of time for its operation, but the results are very complete, a vacuum of $\frac{1}{43000}$ inch of mercury being sometimes obtained. By use of chemicals in addition to the above, a vacuum of $\frac{1}{650000}$ inch of mercury has been obtained.

25. Magdeburg Hemispheres.—The pressure of the atmosphere can be made manifest by means of two hollow hemispheres, such as are shown in Fig. 7. This contrivance was devised by Otto Von Guericke, of Magdeburg, and is known as the **Magdeburg hemispheres.** One of the hemispheres is provided with a stopcock, by which it can be screwed on to an air pump. The edges fit accurately and are well greased, so as to be air-tight. When the hemispheres contain air, they can be separated easily; when the air is pumped out by an air pump, they can be separated only with great difficulty. The force required to separate them will be equal to the area of the largest circle of the hemisphere (projected area) in square inches, multiplied by 14.7 pounds. This force will be the same in whatever position the hemispheres may be held, which proves that the pressure of air on them is the same in all directions.

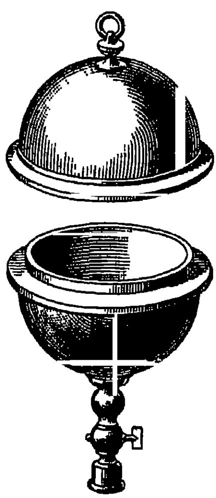


FIG 7

26. The Weight Lifter.—The pressure of the atmosphere is shown by means of the apparatus illustrated in Fig 8. Here, a cylinder fitted with a piston is held in suspension by a chain. At the top of the cylinder is a plug *a*, which can be taken out. This plug is removed and the piston is pushed up until it touches the cylinder head. If the plug is then screwed in, the piston will remain at the top until a weight has been hung on the rod equal to the area of the piston multiplied by 14.7 pounds, less the weight of the piston and rod. If a force is applied to the rod sufficiently great to push the piston downwards, the piston will, on the removal of the force, raise to the top of the cylinder any weight that is less than the one mentioned. Suppose the weight to be removed, and the piston to be supported midway between the top and bottom of the cylinder. Let the plug be removed, air admitted above the piston, and the plug screwed back into its place; if the piston is shoved upwards, the farther up it goes, the greater will be the force necessary to push it, on account of the compression of the air. If the piston is of large diameter, it will also require a great force to pull it out of the cylinder, as a little consideration will show. For example, let the diameter of the piston be 20 inches, the length of the cylinder 36 inches, plus the thickness of the piston, and the weight of the piston and rod 100 pounds. If the piston is in the middle of the cylinder, there will be 18 inches of space above it, and 18 inches of space below it. The area of the piston is $20^2 \times .7854 = 314.16$ square inches, and the atmospheric pressure on it is $314.16 \times 14.7 = 4,618$ pounds, nearly. In order to shove the piston upwards 9 inches, the pressure on it must be twice as great, or 9,236 pounds, and

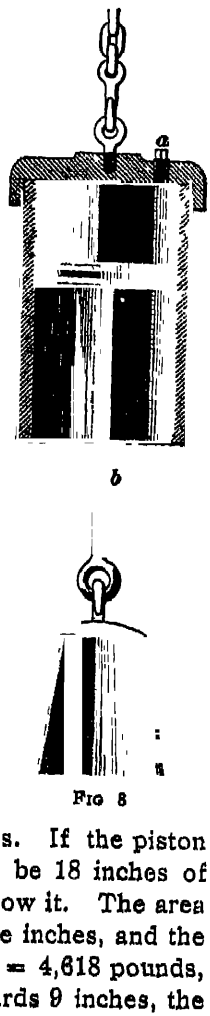


FIG 8

to this must be added 100 pounds, the weight of the piston and rod, which gives 9,236 + 100 = 9,336 pounds. The force necessary to cause the piston to move upwards 9 inches will then be 9,336 − 4,618 = 4,718 pounds. Now, suppose the piston to be moved downwards until it is just on the point of being pulled out of the cylinder The volume above it will then be twice as great as before, and the pressure one-half as great, or 4,618 ÷ 2 = 2,309 pounds. The total upward pressure will be the pressure of the atmosphere less the weight of the piston and rod, or 4,618 − 100 = 4,518 pounds, and the force necessary to pull it downwards to this point will be 4,518 − 2,309 = 2,209 pounds.

27. The Baroscope.—The buoyant effect of air is very clearly shown by means of an instrument called the **baroscope**, shown in Fig. 9. It consists of a scale beam, from one extremity of which is suspended a small weight, and from the other a hollow copper sphere. In air, they exactly balance each other, but when they are placed under the receiver of an air pump and the air is exhausted the sphere sinks, showing that it is really heavier than the small weight. Before the air is exhausted, each body is buoyed up by the weight of the air it displaces, and, since the sphere displaces more air, it loses more weight by reason of this displacement than the small weight. Suppose that the volume of the sphere exceeds that of the weight by 10 cubic inches; the weight of this volume of air is 3.1 grains. If this weight is added to the small weight, it will overbalance the sphere in air, but will exactly balance it in a vacuum.

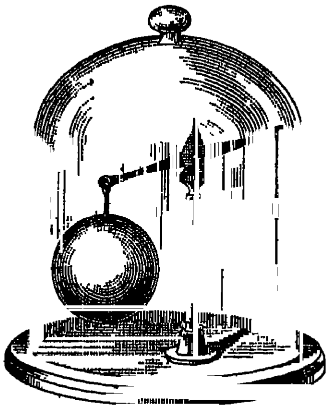


FIG 9

AIR COMPRESSORS

28. For many purposes, compressed air is preferable to steam or other gases for use as a motive power; in such cases, **air compressors** are used to compress the air. These are made in many forms, but the most common one consists of a cylinder, called the *air cylinder*, placed in front of the crosshead of a steam engine, so that the piston of the air cylinder can be driven by attaching its piston rod to the crosshead, in a manner similar to a steam pump. A cross-section of the air cylinder of a compressor of this kind is shown in Fig 10, in which *a* is the piston and *b* is the piston rod, driven by the crosshead of a steam engine not shown in the figure. Both ends of the lower half of the cylinder are fitted with inlet valves *d* and *d'*, which allow the air to enter the cylinder, and both ends of the upper half are fitted with discharge valves *f* and *f'*, which allow the air to escape from the cylinder after it has been compressed to the required pressure.

Suppose the piston *a* to be moving in the direction of the arrow; then the inlet valves *d* in the left-hand end of the cylinder from which the piston is moving will be forced inwards by the pressure of the atmosphere, which overcomes the resistance of the light spring *c*, thus allowing the air to flow in and fill the cylinder. On the other side of the piston, the air is being compressed, and, consequently, it acts with the springs *s* to force the inlet valves *d'* in the right-hand end of the cylinder to their seats. In the right-hand end of the cylinder, the discharge valves *f'* are opened when the pressure of the air in the cylinder is great enough to overcome the resistance of the light springs *e* and the tension of the air in the passages leading to the discharge pipe *h*, and the discharge valves *f* are pressed against their seats by the springs *e* and the tension of the air in the passages. Suppose it is desired to compress the air to 59 pounds per square inch, and to find at what point of the stroke the discharge valves will open. Now, a pressure of 59 pounds per square inch equals a pressure of 4 atmospheres,

very nearly, hence, when the pressure in the cylinder becomes great enough to force air out through the discharge valves, the volume must be one-quarter of the volume at atmospheric pressure, or the valves will open when the piston has traveled three-quarters of its stroke, provided that the air is compressed at constant temperature.

The air, after being discharged from the cylinder, passes out through the delivery pipe h, and from there is con-

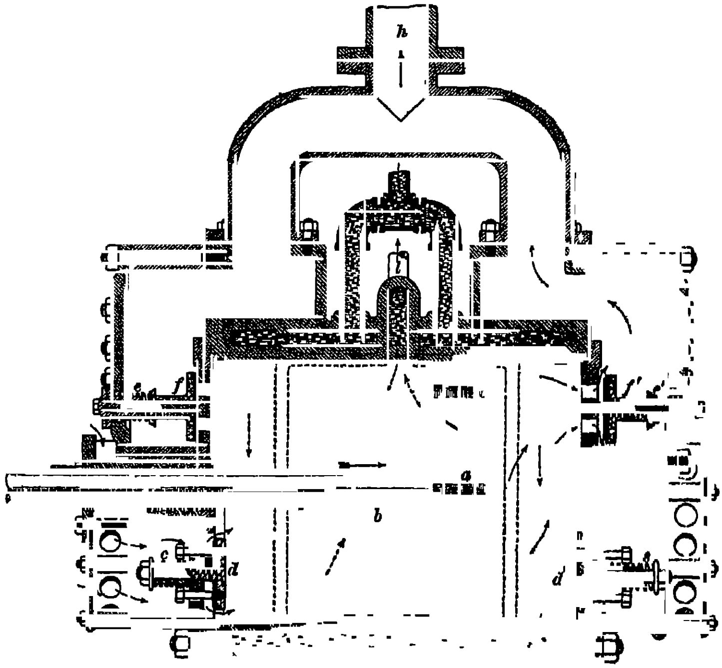


FIG 10

veyed to its destination. It has been shown that when air or any other gas is compressed its temperature is increased. For high pressures, this increase of temperature becomes a serious consideration, for two reasons: (1) When the air is discharged at a high temperature, the pressure falls considerably when the air has cooled down to its normal temperature, and this represents a serious loss in the economical working of the machine. (2) The alternate heating and

cooling of the compressor cylinder by the hot and cold air is very destructive to it, and increases the wear to a great extent. To prevent the air from heating, cooling devices are resorted to, the most common one being the so-called **water-jacket.** This is effected in the following manner: The cylinder walls are hollow, as shown in Fig 10; the cold water enters this hollow space in the cylinder wall through the pipe *k k*, and flows around the cylinder, finally passing out through the discharge pipe *l*. The water keeps cold the cylinder walls, which cool the air as it is compressed.

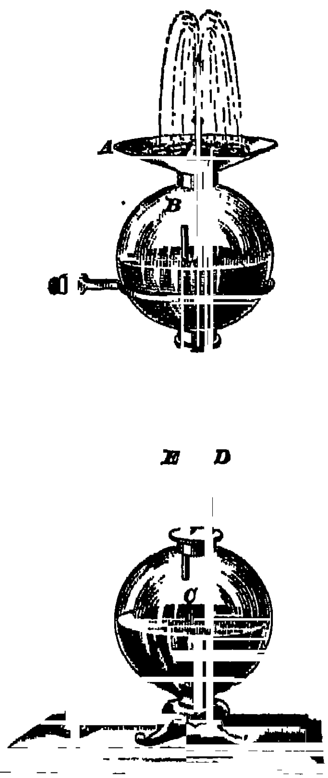


FIG. 11

29. Hero's Fountain. Hero's fountain derives its name from its inventor, Hero, who lived at Alexandria about 120 B. C. This fountain, which is shown in Fig. 11, consists of a brass dish *A* and two glass globes *B* and *C*, and depends for its operation on the elastic properties of air. The dish communicates with the lower part of the globe *C* by a long tube *D*, and another tube *E* connects the two globes. A third tube passes through the dish *A* to the lower part of the globe *B*. This last tube being taken out, the globe *B* is partially filled with water; the tube is then replaced and water is poured into the dish. The water flows through the tube *D* into the lower globe, and expels the air, which is forced into the upper globe. The air thus compressed acts on the water and makes it jet out through the shortest tube, as represented in the figure. Were it not for

the resistance offered by the atmosphere and by friction, the issuing water would rise to a height above the water in the dish equal to the difference of the level of the water in the two globes.

THE SIPHON

30. The action of the siphon illustrates the effect of atmospheric pressure. A siphon is simply a bent tube with unequal branches, open at both ends, and is used to convey a liquid from a higher point to a lower, over an intermediate point higher than either of the other two. In Fig 12, *a* and *b* are two vessels, *b* being lower than *a*, and *a c b* is the bent tube or siphon. Suppose this tube to be filled with water and placed in the vessels, as shown, with the short branch *a c* in the vessel *a*. The water will flow from the vessel *a* into *b*, so long as the level of the water in *b* is below the level of the water in *a* and the level of the water in *a* is above the lower end of the tube *a c*.

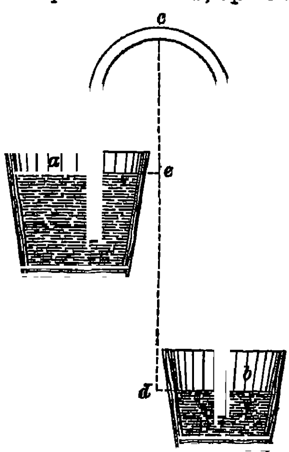


FIG 12

The atmospheric pressure on the surfaces of *a* and *b* tends to force the water up the tubes *a c* and *b c*. When the siphon is filled with water, each of these pressures is counteracted in part by the pressure of the water in that branch of the siphon that is immersed in the water on which the pressure is exerted. The atmospheric pressure opposed to the weight of the longer column of water will, therefore, be more resisted than that opposed to the weight of the shorter column; consequently, the pressure exerted on the shorter column will be greater than that on the longer column, and this excess of pressure will produce motion.

The action of the siphon is of great importance, and should be thoroughly understood. The following considerations and computations will make the subject clear:

Let a = area of tube, in square inches;

$h = dc$ = vertical distance, in inches, between the surface of water in b and highest point of the center line of tube,

$h_1 = ec$ = distance, in inches, between the surface of water in a and highest point of center line of tube.

The weight of the water in the short column is $.03617\,a\,h_1$, and the resultant atmospheric pressure, tending to force the water up the short column, is $14.7 \times a - .03617\,a\,h_1$. The weight of the water in the long column is $.03617\,a\,h$, and the resultant atmospheric pressure, tending to force the water up the long column, is $14.7\,a - .03617\,a\,h$. The difference between these two is $(14.7\,a - .03617\,a\,h_1) - (14.7\,a - .03617\,a\,h) = .03617\,a\,(h - h_1)$. But $h - h_1 = ed$ = difference between the levels of the water in the two vessels.

It will be noticed that the short column must not be higher than 34 feet for water, or the siphon will not work, since the

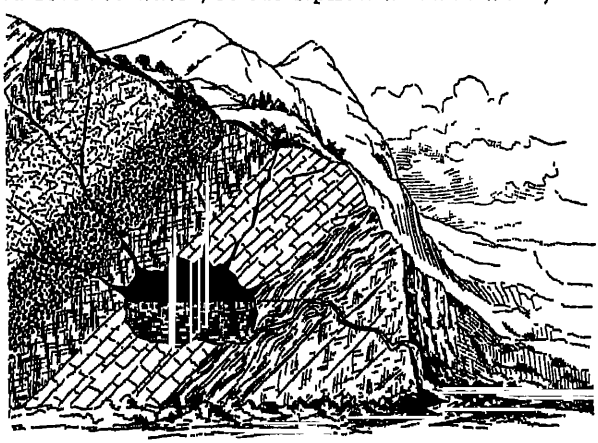


FIG 18

pressure of the atmosphere will not support a column of water that is higher than 34 feet; 28 feet is considered to be the greatest height for which a siphon will work well.

31. Intermittent Springs.—Sometimes a spring is observed to flow for a time and then cease; then, after an

interval, to flow again for a time The generally accepted explanation of this is that there is an underground reservoir fed with water through fissures in the earth, as shown in Fig. 13 The outlet for the water is shaped like a siphon, as shown. When the water in the reservoir reaches the same height as the highest point of outlet, it flows out until the level of the water in the reservoir falls below the mouth of the siphon, if the flow of water is greater than the supply to the reservoir, in which case the flow ceases until the water in the reservoir again reaches the level of the highest point of the siphon.

THE LOCOMOTIVE BLAST

32. Fig. 14 shows the front end of a locomotive: *E* is the exhaust pipe, the center of which is directly in line with

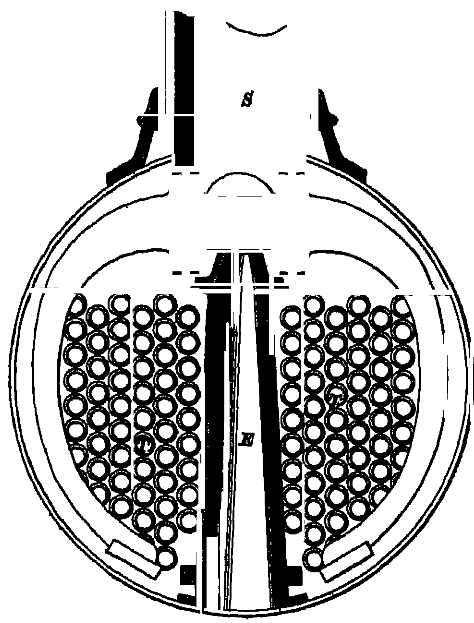


FIG 14

the center of the smokestack *S*, *T*, *T* are the tubes through which the hot furnace gases are discharged. The exhaust

steam has a pressure of about 2 pounds above the atmosphere, and rushes through the exhaust pipe E and up the smokestack S with a very high velocity, taking the air out with it, and producing a partial vacuum in the space in front of the tubes. No air can get in this space except through the grates of the firebox, consequently, the partial vacuum created in front of the tubes causes an influx of air through the grate, and produces the **forced draft,** or **blast.** The faster the engine runs, the greater is the quantity of air drawn through the grate.

PUMPS

33. The Suction Pump.—A section of an ordinary **suction pump** is shown in Fig. 15. Suppose the piston to be at the bottom of the cylinder and to be just on the point of moving upwards in the direction of the arrow. As the piston rises, it leaves a vacuum behind it. The air in P then raises the valve V, and expands in the cylinder B, whereby its pressure is diminished below that of the atmosphere. The atmospheric pressure on the surface of the water in the well

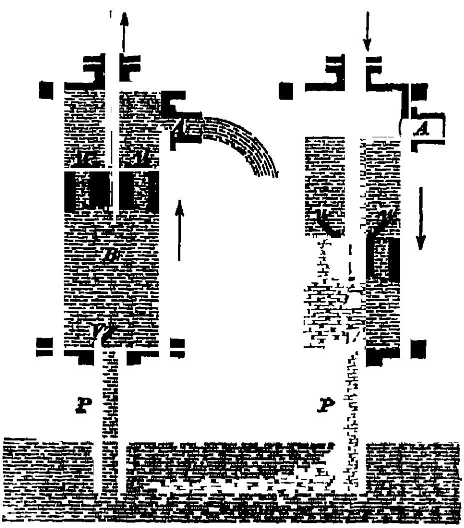


FIG. 15

causes the water to rise in the pipe P. When the piston descends, the air in B escapes through the valves u. After a few strokes, the water fills completely the space under the piston in cylinder B, so that, when the piston reaches the end of its stroke, the water entirely fills the space between the

bottom of the piston and the bottom of the cylinder and also the pipe *P*. The instant that the piston begins its down stroke, the water in the chamber *B* tends to fall back into the well, and its weight forces the valve *V* to its seat, thus preventing any downward flow of the water. The piston now tends to compress the water in the chamber *B*, but this is prevented through the opening of the valves *u*, *u* in the piston. When the piston has reached the end of its downward stroke, the weight of the water above closes the valves *u*, *u*. All the water resting on the top of the piston is then lifted with the piston on its upward stroke, and discharged through the spout *A*, the valve *V* again opening, and the water filling the space below the piston as before

It is evident that the distance between the valve *V* and the surface of the water in the well must not exceed 34 feet, the highest column of water that the pressure of the atmosphere will sustain, since otherwise the water in the pipe would not reach to the height of the valve *V* In practice, this distance should not exceed 28 feet This is due to the fact that there is a little air left between the bottom of the piston and the bottom of the cylinder, a little air leaks through the valves, which are not perfectly air-tight, and a pressure is needed to raise the valve against its weight, which, of course, acts downwards. There are many varieties of the suction pump, differing principally in the valves and piston, but the principle is the same in all.

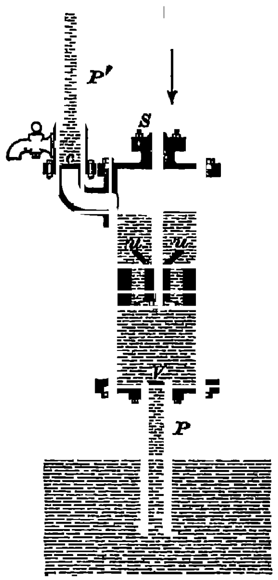


FIG 16

34. The Lifting Pump.—A section of a **lifting pump** is shown in Fig 16 These pumps are used when water is to be raised to greater heights than can be done with the ordinary suction pump. As will be perceived, it is essentially the same as the suction

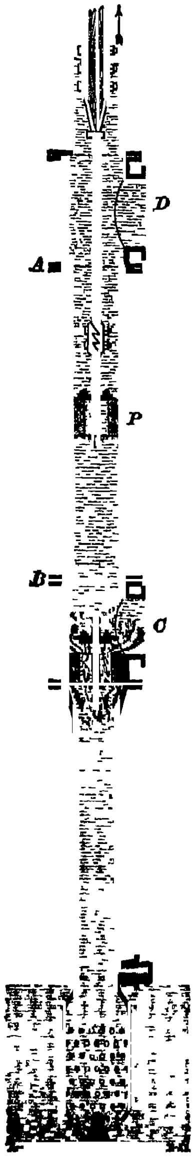


FIG. 17

pump, except that the spout is fitted with a cock and has a pipe attached to it, leading to the point of discharge. If it is desired to discharge the water at the spout, the cock may be opened, otherwise, the cock is closed, and the water is lifted by the piston up through the pipe P' to the point of discharge, the valve c preventing it from falling back into the pump, and the valve V preventing the water in the pump from falling back into the well. It is not necessary that there should be a second pipe P', as shown in the figure, for the pipe P may be continued straight upwards, as shown in Fig 17. This figure shows a section of a lifting pump for raising water from great depths, as from the bottom of mines to the surface. The pump consists of a series of pipes connected together, of which the lower end only is shown in the figure. That part of the pipe included between the letters A and B forms the pump cylinder, in which the piston P works. That part of the pipe above the highest point of the piston travel, through which the water is discharged, is called the **delivery pipe**, and the part below the lowest point of the piston travel is called the **suction pipe**. The lower end of the suction pipe is expanded, and has a number of small holes in it, to keep out solid matter. C is a plate covering an opening, and may be removed to allow the suction valve to be repaired. D is a plate covering a similar opening, through which the piston and piston valves may be repaired. The piston rod, or rather the piston stem, is made of wrought iron, inserted with wood, and connected with the piston. The only limit to the height to which a pump of this

kind can raise water is the strength of the piston rod Lifting pumps of this kind are used to raise water from great depths to the earth's surface; hence, a very long piston rod is necessary. In the lifting pump shown in Fig. 16, the water is raised from a point a few feet below the earth's surface to a point considerably higher. This requires the piston rod to move through a stuffingbox, as shown at *S*, and also necessitates the rod being round, in order that the water may not leak out.

35. Force Pumps.—The **force pump** differs from the lifting pump in several important particulars, but chiefly in the fact that the piston is solid, that is, it has no valves A section of a *suction and force pump* is shown in Fig. 18. The water is drawn up the suction pipe as before, when the piston rises, but when the piston reverses the pressure on the water caused by the descent of the piston opens the valve V' and forces the water up the delivery pipe P'. When the piston again begins its upward movement, the valve V' is closed by the pressure of the water above it, and the valve V is opened by the pressure of the atmosphere on the water below it, as in the previous cases. For an arrangement of this kind, it is not necessary to have a stuffingbox. The water may be forced to almost any desired height. The force pump differs again from the lifting pump in respect to its piston rod, which should not be longer than is absolutely necessary in order to prevent it from buckling, while, in the lifting pump, the length of the piston rod is a matter of indifference.

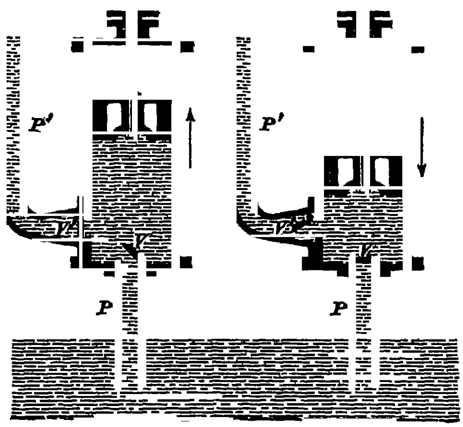


FIG 18

pressure below the plunger being less than the pressure of the atmosphere above, the air would rush in instead of being expelled.

37. Double-Acting Pumps.—In the pumps previously described, the discharge was intermittent, that is, the pump could only discharge when the piston was moving in one direction. In some cases, it is necessary that there should

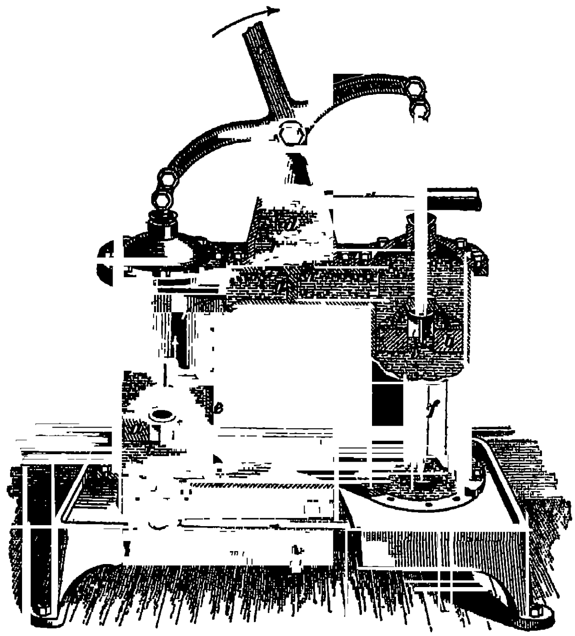


FIG 20

be a continuous discharge; in all cases, it takes more power to run the pump with an intermittent discharge, as a little consideration will show. If the height that the water is to be raised is considerable, its weight will be very great, and the entire mass must be put in motion during one stroke of the piston.

In order to obtain the advantage of a more continuous discharge, double-acting pumps are used. Fig 20 shows a part sectional view of such a pump. Two pistons *a* and *b*

are used, which are operated by one handle *c* in the manner shown. The pump has one suction pipe *s* and one discharge pipe *d*. The cylinders *e* and *f* are separated by a diaphragm *g*, so that they cannot communicate with each other above the pistons. In the figure, the handle *c* is moving to the right, the piston *a* upwards, and the piston *b* downwards. In moving upwards, the piston *a* lifts the water above it, causing it to flow through the delivery valve *h* into the discharge pipe *d*. This upward movement of the piston creates a partial vacuum below it in the cylinder *e*, and causes the water to rush up the suction pipe *s* into the cylinder, as shown by the arrows. In the cylinder *f*, the downward movement of the piston *b* raises the piston valve *v*, and the weight of the water on the suction valve *i* keeps it closed. When the handle *c* has completed its movement to the right and begins its return, all the valves on the right-hand side open except *v*, and those on the left-hand side close except *t*; water is then discharged into the delivery pipe by the cylinder *f*, and only at the instant of reversal is the flow into the delivery pipe *d* stopped.

38. Air Chambers.—In order to obtain a continuous flow of water in the delivery pipe, with as nearly a uniform velocity as possible, an **air chamber** is usually placed on the delivery pipe of force pumps as near to the pump cylinder as the construction of the machine will allow. The air chambers are usually pear-shaped, with the small end connected to the pipe. They are filled with air, which the water compresses during the discharge. During the suction, the air thus compressed expands and acts as an accelerating force on the moving column of water, a force that diminishes with the expansion of the air, and helps to keep the velocity of the moving column more nearly uniform. An air chamber is sometimes placed on the suction pipe. These air chambers not only tend to promote a uniform discharge, but they also equalize the stresses on the pump, and prevent shocks due to the incompressibility of water. They serve the same purpose in pumps that flywheels do in steam engines. Unless

the pump moves very slowly, it is absolutely necessary to have an air chamber on the delivery pipe.

39. Steam Pumps.—Steam pumps are force pumps operated by steam acting on the piston of a steam engine, directly connected to the pump, and in many cases cast with the pump. A section of a double-acting steam pump showing the steam and water cylinders, with other details, is illustrated in Fig. 21. Here G is a steam piston, and R the piston rod, which is secured at its other end to the plunger P. F is a partition cast with the cylinder, which prevents the water in

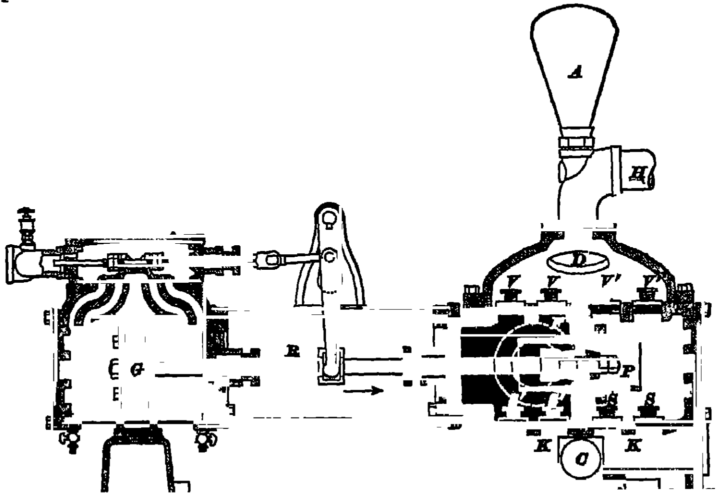


FIG 21

the left-hand half from communicating with that in the right-hand half of the cylinder. Suppose the piston to be moving in the direction of the arrow The volume of the left-hand half of the pump cylinder will be increased by an amount equal to the area of the circumference of the plunger multiplied by the length of the stroke, and the volume of the right-hand half of the cylinder will be diminished by a like amount In consequence of this, a volume of water in the right-hand half of the cylinder equal to the volume displaced by the plunger in its forward motion will be forced through the valves V', V' into the air chamber A, through the orifice D,

and then discharged through the delivery pipe *H*. By reason of the partial vacuum in the left-hand half of the pump cylinder, owing to this movement of the plunger, the water will be drawn from the reservoir through the suction pipe *C* into the chamber *K*, *K*, lifting the valves *S'*, *S'*, and filling the space displaced by the plunger. During the return stroke, the water will be drawn through the valves *S*, *S* into the right-hand half of the pump cylinder, and discharged through the valves *V*, *V* in the left-hand half. Each of the four suction and four discharge valves is kept to its seat, when not working, by light springs, as shown.

There are many varieties and makes of steam pumps, the majority of which are double-acting. In many cases, two steam pumps are placed side by side, having a common delivery pipe. This arrangement is called a **duplex pump.** It is usual so to set the steam pistons of duplex pumps that when one is completing the stroke the other is in the middle of its stroke A double-acting duplex pump made to run in this manner, and having an air chamber of sufficient size, will deliver water with a nearly uniform velocity

In mine pumps for forcing water to great heights, the plungers are made solid, and in most cases are extended through the pump cylinder. In many steam pumps, pistons are used instead of plungers, but when very heavy duty is required plungers are preferred.

40. Centrifugal Pumps.—Next to the direct-acting steam pump, the **centrifugal pump** is the most valuable instrument for raising water to great heights. As the name implies, the effects produced by centrifugal force are made use of. Fig. 22 represents a centrifugal pump with half of the casing removed. The hub *S* is hollow, and is connected directly to the suction pipe. The curved arms *a*, called **vanes** or **wings**, are revolved with a high velocity in the direction of the arrow, and the air enclosed between them is driven out through the discharge passage and delivery pipe *D D*. This creates a partial vacuum in the casing and suction pipe, and causes the water to flow in through *S*. This water is also

made to revolve with the vanes, and, of course, with the same velocity. The centrifugal force of the revolving water causes it to fly outwards toward the end of the vanes, and becomes greater the farther away the water gets from the center. This causes the water to leave the vanes, and finally to leave the pump by means of the discharge passage and delivery pipe $D D$. The height to which the water can be forced depends on the velocity of the revolving vanes. In the construction of a centrifugal pump, particular care is required in giving the correct form to the vanes, for the efficiency of the machine depends greatly on this feature. What is required is to raise the water, and the energy used to drive the pump should be devoted as much as possible to this one purpose. The water, when it is raised, should be delivered with as little velocity as possible, for any velocity that the water then possesses has been secured at the expense of the energy used to drive the pump. The form of the vanes is such that the water is delivered at the desired height with the least expenditure of energy.

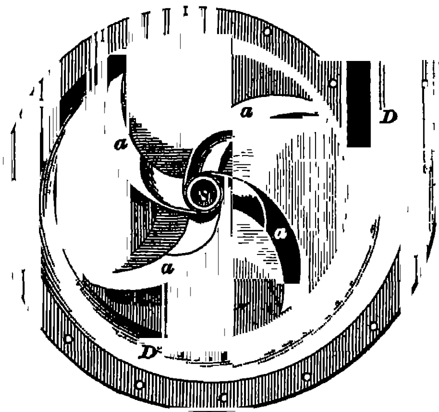


FIG 22

The number of vanes depends on the size and capacity of the pump. It will be noticed that, in the pump shown in the figure, the vanes have sharp edges near the hub. The object of this is to provide for a free ingress of the water, and also to cut any foreign substance that may enter the pump and prevent it from working properly.

Almost any liquid can be raised with these pumps, but, when they are intended for pumping chemicals, the casing and vanes should be made of materials that will not be acted on by the chemicals.

41. The Hydraulic Ram.—The construction of a hydraulic ram is shown in Fig. 23. This machine is used for raising water from a point below the level of the water in a spring or reservoir to a point considerably higher, with no power other than that afforded by the inertia of a moving column of water. In the figure, *a* is a pipe called the *drive pipe*, connecting the ram with the reservoir; the valve *b* slides freely in a guide, and is provided with locknuts to regulate the distance that it can fall below its seat. When the water is first turned on by opening the valve *n*, the valve *b* is already opened, and the water flows out through *c*, as shown.

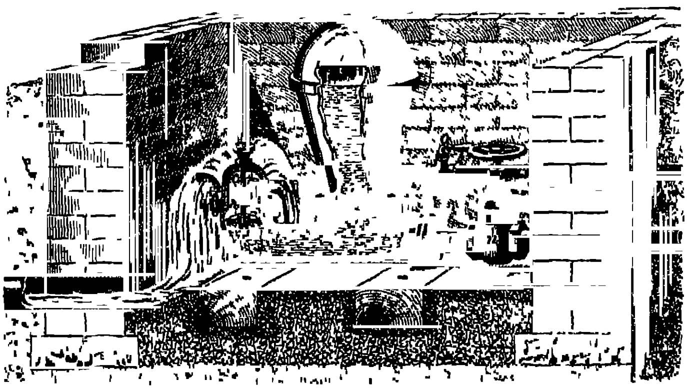


FIG. 23

As the discharge continues, the velocity of the water in the drive pipe will increase until the upward pressure against the valve *b* is sufficient to force the valve to its seat. The actual closing of the valve takes place very suddenly, and the momentum of the column of water, which was moving with an increasing velocity through the drive pipe *a*, will very rapidly force some water through the valve *d* into the air chamber *f*. Immediately after this, a rebound takes place, and for a short interval of time the water flows back up the drive pipe *a* and tends to form a vacuum under the air-chamber valve *d*; this opens the snifter valve *g* and admits a little air, which accumulates under the valve *d* and is forced into the air chamber with the next shock. This air keeps the

air chamber constantly charged, otherwise, the water, being under a greater pressure in the air chamber than in the reservoir, would soon absorb the air in the chamber and the ram would cease to work until the chamber was recharged with air The rebound also takes the pressure off the under side of the valve *b* and causes it to drop, and the above-described operations are repeated. The delivery pipe is shown at *e*; a steady flow of water is maintained through it by the pressure of the air in the chamber *f*; this air also acts as a cushion when valve *b* suddenly closes, and prevents undue shock to the parts of the ram.

The height to which water can be raised by the hydraulic ram depends on the weight of the valve *b* and the velocity of the water in *a*.

42. Power Necessary to Work a Pump.—

Principle I.—*In all pumps, whether lifting, force, steam single- or double-acting, or centrifugal, the number of foot-pounds of power needed to work the pump is equal to the weight of the water in pounds, multiplied by the vertical distance, in feet, between the level of the water in the well, or source, and the point of discharge, plus the work necessary to overcome the friction and other resistances.*

Principle II.—*The work done in one stroke of a pump is equal to the weight of a volume of water equal to the volume displaced by the piston during the stroke, multiplied by the total vertical distance, in feet, through which the water is to be raised, plus the work necessary to overcome the resistances.*

A little consideration will make Principle II evident. Suppose that the height of the suction is 25 feet; that the vertical distance between the suction valve and the point of discharge is 100 feet; that the stroke of the piston is 15 inches, and that its diameter is 10 inches. Let the diameters of the suction pipe and delivery pipe be 4 inches each. The volume displaced by the pump piston or plunger in one stroke equals $\frac{10^2 \times .7854 \times 15}{1,728} = .68177$ cubic foot. The weight of an equal volume of water is $.68177 \times 62.5$

= 42 611 pounds Now, in order to discharge this water, *all* the water in the suction and delivery pipes has to be moved through a certain distance, in feet, equal to 68177 divided by the area of the pipes, in square feet.

4 inches = $\frac{1}{3}$ foot. $(\frac{1}{3})^2 \times .7854 = \frac{.7854}{9} = .0872\frac{2}{3}$ square foot. $.68177 \div .0872\frac{2}{3} = 7\ 8125$ feet.

The weight of water in the delivery pipe is $(\frac{1}{3})^2 \times .7854 \times 100 \times 62.5 = 545\ 42$ pounds.

The weight of water in the suction pipe is $(\frac{1}{3})^2 \times .7854 \times 25 \times 62\ 5 = 136.35$ pounds.

$545\ 42 + 136\ 35 = 681\ 77$ pounds, which is the total weight of water moved in one stroke. The distance that the water is moved in one stroke is 7.8125 feet; hence, the number of foot-pounds necessary for one stroke is $681.77 \times 7.8125 = 5{,}326\ 3$ foot-pounds. Had this result been obtained by Principle II, the process would have been as follows: The weight of the water displaced by the piston in one stroke was found to be 42.611 pounds. $42.611 \times 125 = 5{,}326.4$ pounds, which is practically the same as the result obtained by the previous method, and is a great deal shorter. The slight difference between the two results is due to neglected decimals.

EXAMPLE.—What must be the necessary horsepower of a double-acting steam pump if the vertical distance between the point of discharge and the point of suction is 96 feet? The diameter of the pump cylinder is 8 inches, the stroke is 10 inches, and the number of strokes per minute is 120. Allow 25 per cent. for friction and other resistances.

SOLUTION.—Since the pump is double-acting, it raises a quantity of water equal to the volume displaced by the plunger at every stroke. The weight of the volume of water displaced at one stroke is $(\frac{8}{12})^2 \times .7854 \times \frac{10}{12} \times 62.5 = 18\ 18$ lb., nearly.

$18.18 \times 96 \times 120 = 209{,}430$ ft.-lb. per minute.

Since 25 per cent. is to be allowed for friction, the actual number of foot-pounds per minute is $209{,}430 \div .75 = 279{,}240$. 1 H. P. = 33,000 ft.-lb. per min.; hence, $\frac{279{,}240}{33{,}000} = 8.462$ H. P., nearly. Ans.

If the acceleration a is supposed to have a fixed value, as 8 feet per second, and different values are assigned to t, different values will be obtained for s. Here, too, t and s are variables t, which is varied at pleasure, is the independent variable; and s, whose values depend on those of t, is the dependent variable, or a function of t. The acceleration a, although represented by a letter, is assumed to be fixed or invariable, and is therefore a constant.

2. In general, when, for any particular purpose, some of the quantities represented by letters in an equation are made to take (or are considered as being such that they can take) different values, they are called **variables**. Those to which values are assigned arbitrarily are called **independent variables**; those whose values depend on the values of the independent variables are called **dependent variables**, or **functions** of the independent variables. Those quantities that are supposed to remain fixed are called **constants**.

A function may also be defined as a quantity whose value depends on the value or values of one or more other quantities; the very word "depends" indicates that the function can have different values (that is, can *vary*) according to the values assigned to other quantities. Thus, the area of a triangle is a function of the base and altitude; if the base is assumed to be fixed, it becomes a constant, and the area is a function of only the altitude. The velocity of a body moving under the action of an unbalanced force is a function of the magnitude of the force and the mass of the body; if the mass is assumed to be fixed, it becomes a constant, and the velocity is a function of only the force.

3. Graph of an Equation.—It was stated in Art. 1 that from the equation $A = \pi r^2$, a table can be made giving the values of the function A corresponding to different values of the independent variable r. Instead of a table, a diagram may be constructed to represent the relation between the values of A and r. Such a diagram, which is the graphic representation of the equation just given, is called the **graph** of that equation, and is constructed as follows:

value of r is laid off along OX from O to, say, M, a perpendicular is erected at M, intersecting the graph at P. Then MP will represent the value of A corresponding to the value OM of r.

4. The perpendicular distances of any point of the graph from the axes are called the **coordinates** of that point. Thus, the coordinates of P are MP ($= OM'$) and $M'P$

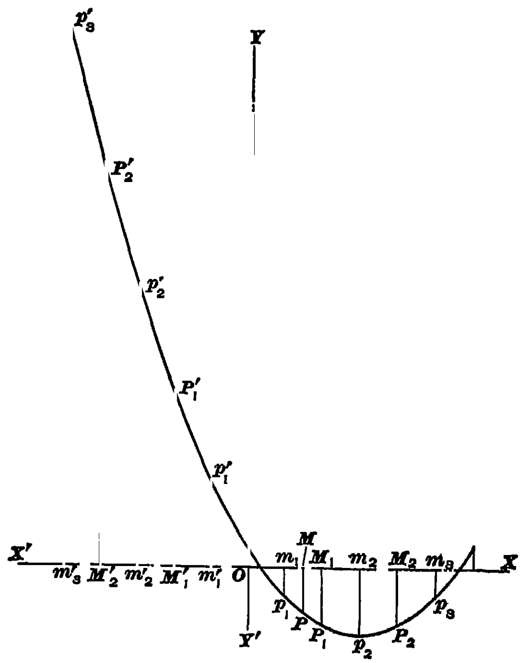


FIG 2

($= OM$). The horizontal coordinate OM, or, more generally, the coordinate representing the independent variable, is usually called the **abscissa**, and the other coordinate OM', the **ordinate**. It is customary to reckon the abscissa along the axis OX, called the **axis of x**, and the ordinate on a line parallel to OY, through the foot of the abscissa. Thus, the coordinates of P are stated as OM and MP, instead of $M'P$ and MP.

connects with the crank OK by the connecting-rod HK. It is shown in mechanics that when the crank has described an angle X from the position OA, which is in line with the axis of the cylinder, the velocity u of the piston is given approximately by the formula

$$u = v\left(\sin X + \frac{a \sin 2X}{2l}\right)$$

in which v = linear velocity of crankpin K;
l = length of connecting-rod;
a = length of crank.

If the ratio $\frac{a}{l}$ is represented by c, the formula may be written,

$$\frac{u}{v} = \sin X + \frac{c}{2}\sin 2X \qquad (1)$$

When c is given, the graph of equation (1) can be constructed by taking X as the independent variable, and $\frac{u}{v}$ as

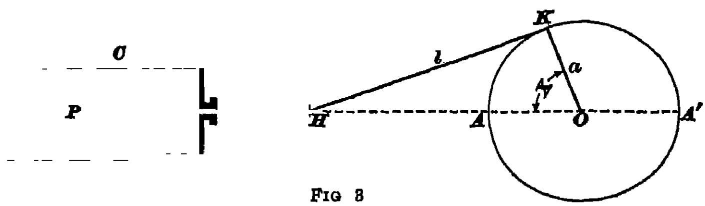


FIG 3

the function. Then, the value of $\frac{u}{v}$ for any value of X can be found from the graph, and, when v is given, u can be determined by multiplying by v the value found for $\frac{u}{v}$.

EXAMPLE 1.—To construct the graph of equation (1) when $c = \frac{1}{5}$, for values of X varying from 0° to 180°, that is, for one-half a revolution of the crank, from the position OA to the position OA', Fig. 3.

SOLUTION.—Writing, for shortness, y for $\frac{u}{v}$, and substituting the given value of c, equation (1) becomes

$$y = \sin X + \tfrac{1}{10}\sin 2X$$

By giving to X values from 0° to 180°, at intervals of 6°, the following table is obtained:

EXAMPLES FOR PRACTICE

1 Plat the following equations for values of x varying from -3 to 8 (*a*) $y = 3x^3 - 7x^2 + 4x - 5$ (*b*) $y = 4x^2 - 10x + 4$ (*c*) $y = (3 + x)^2 - 12$

$$\text{Aus}\begin{cases}(a)\ \text{The general form of the graph is shown in Fig 5}\\(b)\ \text{The general form of the graph is shown in Fig 6}\\(c)\ \text{The general form of the graph is shown in Fig 7}\end{cases}$$

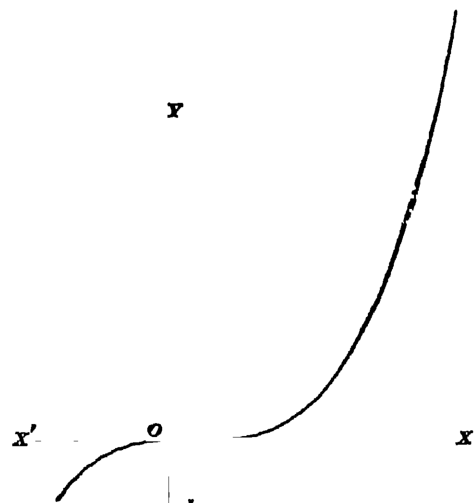


FIG 5

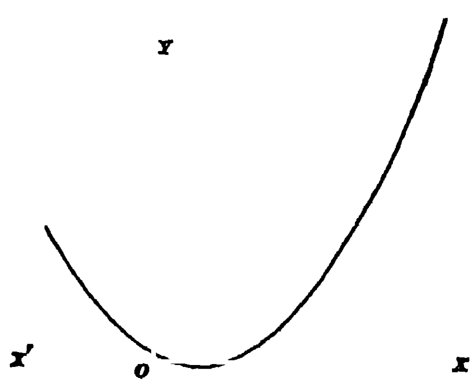


FIG. 6

have been proposed to determine the relation between these two quantities, no exact formula has yet been found. Suppose that a series of experiments on wrought-iron columns gives the following results:

Ratios of Length to Diameter	Breaking Load Pounds per Square Inch
6	51,200
7	47,400
8	44,600
9	42,200
10	40,200
11	38,700
12	37,900
13	37,100
14	36,900

These results may be represented graphically, as shown in Fig 9. Having drawn two coordinate axes OX and OY,

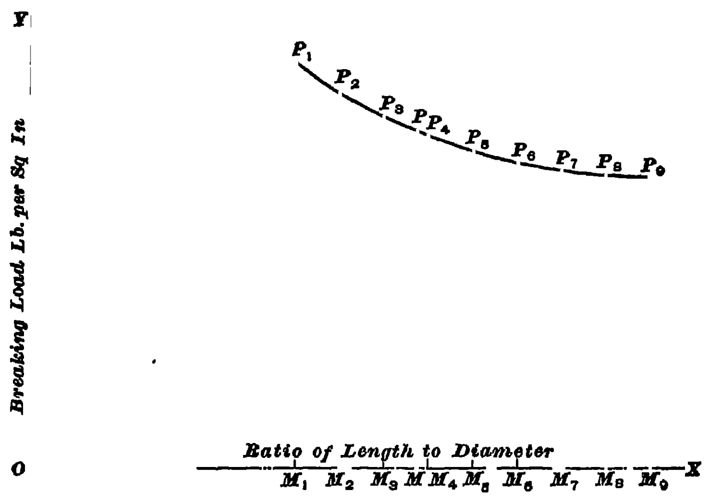


Fig. 9

the values of the ratio of length to diameter, which is the independent variable, are laid off from O, along OX, to any convenient scale. Thus, OM_1 represents the ratio 6; OM_2 represents 7; OM_3, 8; etc. The corresponding values

values written on the respective projections. Thus, the point P_3 is projected at M_3', on which is written 31,500,000, the population in 1860.

The probable population in any intermediate year can be readily obtained from the curve. For instance, if it is desired to find the population in 1878, it will be observed that 1878 lies between 1870 and 1880, and that the interval between 1870 and 1878 is .8 of the interval between 1870 and 1880. Therefore, the required population is found by laying off M_4M equal to .8 of M_4M_5, and drawing the ordinate MP, which represents the approximate population in 1878.

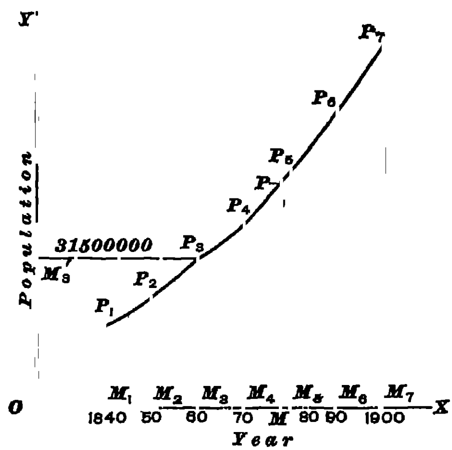


FIG 10

8. In some cases, a diagram is constructed rather for the purpose of presenting to the eye, in a striking manner, the variations of certain quantities, than for the purpose of determining intermediate values, the variations in the latter being too irregular. In the examples so far given, the curves are fairly regular, which shows that the variations in the functions (ordinates) are not too abrupt, and that the curves may be depended on to give tolerably approximate values of the function corresponding to intermediate values of the independent variable Example 1 of the following Examples for Practice is a case in which the curve, on account of its too great irregularity, could not be depended on to give intermediate values. In such cases, the extremities of the ordinates are joined by straight lines, instead of by a curve.

EXAMPLES FOR PRACTICE

1 Draw a graph representing the average daily water consumption in the city of Brooklyn between January, 1871, and December, 1873, from the following data.

Year and Month		Average Daily Consumption, Gallons
1871	January	21,000,000
	April	17,500,000
	July	19,500,000
	October	18,500,000
1872	January	23,500,000
	April	20,500,000
	July	22,500,000
	October	23,000,000
1873	January	29,000,000
	April	22,500,000
	July	26,500,000
	October	22,500,000
	December	24,500,000

Ans. The curve is shown in Fig. 11

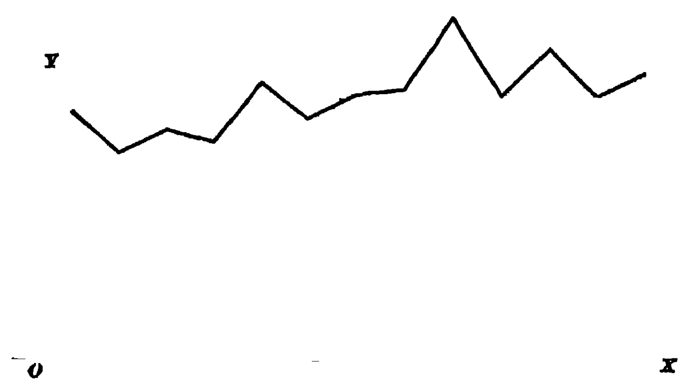


FIG. 11

2. (*a*) Taking the hours as abscissas, and the discharges as ordinates, construct a graph for the water discharged by a pipe from the following observed values for 1 day, (*b*) determine, by means of the curve, the probable discharge at 11:30 A. M.; (*c*) determine the probable discharge at 5:30 P. M.

SUGGESTION.—The scale of ordinates should be chosen sufficiently large to show the differences in discharge—say 4 inches to 1 cubic foot.

HOUR A. M.	DISCHARGE CUBIC FEET PER SECOND	HOUR A M	DISCHARGE CUBIC FEET PER SECOND
1	2.45	11	2.54
2	2 67	M.	
3	2 69	12	2 33
4	2.52	P M.	
5	2.36	1	2 64
6	2.30	2	2 15
7	2.23	3	1 19
8	2.19	4	1 19
9	2 20	5	2 25
10	2.00	6	2 37

FIG 12

Ans. (a) The general form of the graph is shown in Fig. 12
(b) 2 43 cu. ft. per sec.
(c) 2 31 cu. ft per sec.

EQUATIONS OF LINES

INTRODUCTION

9. Equation of a Line.—As already explained, a graph is constructed from an equation expressing a relation between two variables, or giving the value of one variable as a function of the other. In the graph, these variables are the coordinates of points on a line, usually curved. Sometimes, on the contrary, a line is given, and it is required to find the equation of which the line is the graph. That equation is called the equation of the given line, and is a general expression of the relation between the two coordinates of any point of the line, with reference to two coordinate axes conveniently chosen.

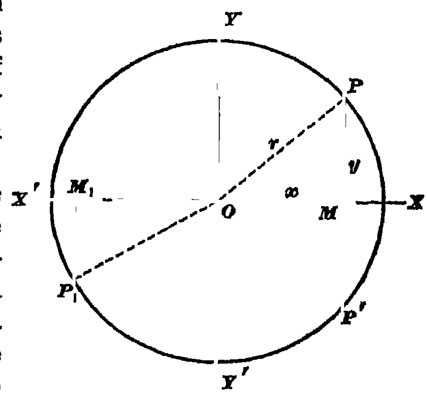


FIG. 13

Take, for instance, a circle of radius r, Fig. 13, and two rectangular axes OX, OY, passing through its center. Let P be any point on the circumference, and x and y its coordinates, as shown. The right triangle OMP gives

$$\overline{OM}^2 + \overline{MP}^2 = \overline{OP}^2;$$

that is,

$$x^2 + y^2 = r^2 \qquad (1)$$

This is the equation of the circle referred to two rectangular axes through the center. In that equation, x and y are the coordinates of any point on the circumference. No

matter where the point is located, its coordinates satisfy equation (1). Thus, for the point P_1,

$$x = -OM_1,\ y = -M_1P_1,$$

and

$$x^2+y^2 = (-OM_1)^2+(-M_1P_1)^2 = OM_1^2+M_1P_1^2$$
$$= OP_1^2 = r^2$$

10. The equation of a line is useful in the study of the geometric properties of the line, and it often serves to recognize the form of a graph corresponding to a given equation If, for example, it is found in the solution of a mechanical problem that the path AB, Fig. 14, of a moving point is such that the coordinates x and y of any of its points, with regard to the axes OX and OY, are related by the equation

$$x^2 + y^2 = a^2$$

the quantity a being constant, it can be at once concluded that the path of the point is a circle whose center is O and whose radius is a.

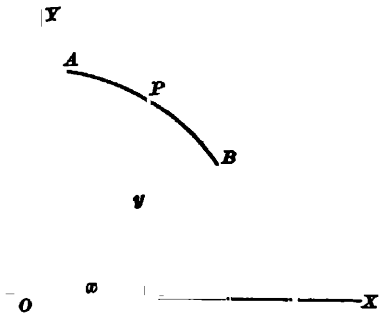


FIG 14

11. **Analytic geometry** is that branch of mathematics in which geometric figures are studied by means of their equations. Surfaces, as well as plane lines, have equations; but in this Course only a few plane lines will be treated.

THE STRAIGHT LINE

12. Equation of the Straight Line.—Let $X'X$ and $Y'Y$, Fig. 15, be two axes of coordinates, and AB a straight line making with $X'X$ an angle H, and intersecting $Y'Y$ at I, the distance b $(= OI)$ being known. It should be understood that the angle H is always measured from the axis $X'X$ upwards. Thus, if the line were A_1B_1, the angle H would be XJ_1B_1. Also, b, like the ordinates, is

positive upwards and negative downwards. Thus, for the line $A_1 B_1$, the value of b is $-O I_1$ This being understood, the equation now to be derived is entirely general, and, with the symbols interpreted as just explained, applies to all cases.

The right triangle PMJ gives

$$PM = JM \tan H;$$

that is,

$$y = (x + JO) \tan H.$$

Now,

$$JO = b \cot H = \frac{b}{\tan H};$$

therefore,

$$y = \left(x + \frac{b}{\tan H}\right) \tan H = x \tan H + b$$

This is the equation of the line AB. It is customary to denote $\tan H$ by a, and write the equation in the form

$$y = ax + b$$

The distances OI and OJ are called the **intercepts** of AB on the axes of y and x, respectively. It is evident that, at I, the abscissa x is 0, and the ordinate is OI; and at J the ordinate y is 0, and the abscissa is $-OJ$.

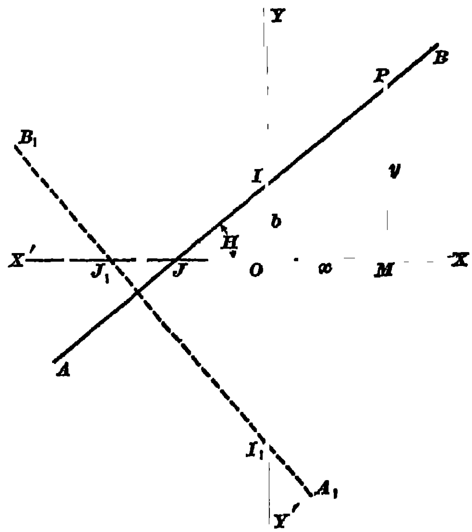


FIG. 15

13. Graph of Any Equation of the First Degree. Any equation of the first degree between two variables can be represented by a straight line; in other words, the graph of any equation of the first degree between two variables is a straight line.

Let

$$mx + ny = p \qquad (1)$$

be any equation of the first degree between the variables

x and y, the other quantities m, n, p being constants, either positive or negative. Solving the equation for y,

$$y = -\frac{m}{n}x + \frac{p}{n} \qquad (2)$$

Since the tangents of the angles between 0° and 180° contain all possible numbers, both positive and negative, it is always possible to find an angle whose tangent is $-\frac{m}{n}$. Let that angle be denoted by H, and denote $\frac{p}{n}$ by b. Then, $-\frac{m}{n} = \tan H$, and equation (2) becomes

$$y = x \tan H + b \qquad (3)$$

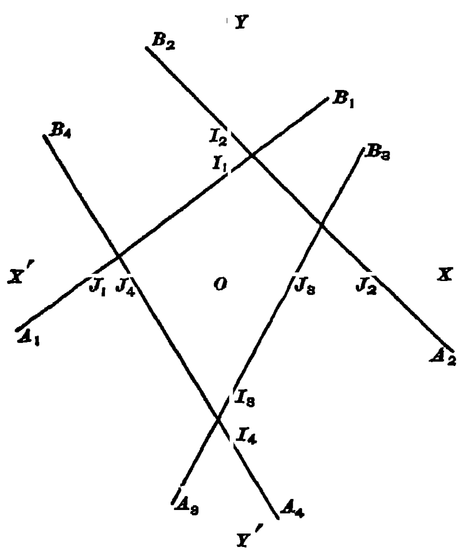


Fig 16

If $-\frac{m}{n}$ is positive, H is less than 90°; if negative, H is greater than 90°.

If on the y axis the distance b is laid off from O, upwards if b is positive, downwards if b is negative, and from the extremity of b a line is drawn making with the x axis an angle equal to H, that line is the graph of equation (3), and, therefore, also of equation (1). If b and $\tan H$ are positive, the line will have such a position as A_1B_1, Fig 16, in which $OI_1 = b$, and $XJ_1B_1 = H$. If b is positive and $\tan H$ negative, the graph will have a position like A_2B_2, in which $OI_2 = b$, and $XJ_2B_2 = H$. If b is negative and $\tan H$ positive, the graph will be like A_3B_3, in which $OI_3 = b$, and $XJ_3B_3 = H$. If both b and $\tan H$ are negative, the graph will have such a position as A_4B_4.

14. Because any equation of the first degree between two variables can be represented by a straight line, formulas in which the value of a quantity is given in terms of the first power of another are called **straight-line formulas.** For example, the formula

$$p = 10{,}000 - 45\frac{l}{r}$$

which expresses the intensity p of pressure that a column can stand, in terms of the ratio $\frac{l}{r}$ of length to radius, is a straight-line formula. If y is written instead of p, and x instead of $\frac{l}{r}$, the formula becomes

$$y = 10{,}000 - 45x$$
$$= -45x + 10{,}000$$

which is the standard form of the equation of a straight line.

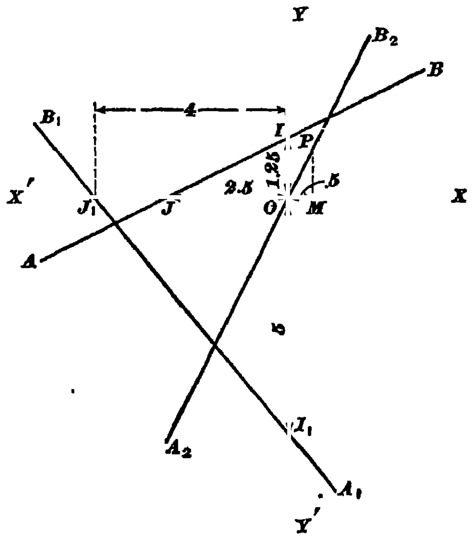


FIG 17

15. The simplest way to draw the straight line corresponding to an equation of the first degree is as follows: Let the equation be $mx + ny + p = 0$.

Select the axes in any convenient position, as $X'X$ and $Y'Y$, Fig. 17. Making $x = 0$ in the equation, and solving for y, the intercept (say OI_1) on the y axis is obtained, and the point I_1 where the line intersects that axis is determined. Making $y = 0$, and solving for x, the intercept OJ_1 and the point J_1 are obtained. The line is then drawn through the points I_1 and J_1.

EXAMPLE 1.—To draw the graph of the equation $8x - 16y + 20 = 0$.

SOLUTION.—Draw the axes $X'X$, $Y'Y$, Fig. 17 Making $x = 0$ in the equation, we have $-16y + 20 = 0$; whence

$$y = OI = 1.25$$

Making $y = 0$ in the equation, we have $8x + 20 = 0$, whence

$$x = OJ = -2.5$$

Laying off $OI = 1.25$, and $OJ = 2.5$, the line AB drawn through I and J is the required graph Ans

EXAMPLE 2 —To draw the graph of the equation $10x + 8y + 40 = 0$.

SOLUTION —Making $x = 0$, and solving for y, Fig. 17,

$$y = OI_1 = -\frac{40}{8} = -5$$

Making $y = 0$, and solving for x,

$$x = OJ_1 = -\frac{40}{10} = -4$$

Laying off $OJ_1 = 4$, $OI_1 = 5$, and joining I_1 and J_1, the required graph A_1B_1 is obtained Ans.

EXAMPLE 3 —To draw the graph of the equation $4y - 8x = 0$.

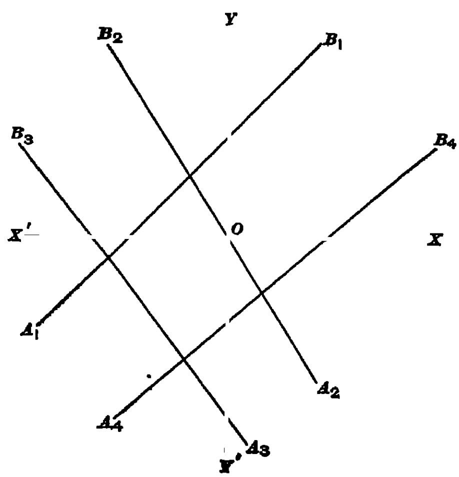


FIG 18

SOLUTION —It will be noticed that this equation has no term independent of x and y. This shows that the line passes through the origin of coordinates, or that its intercepts on the two axes are zero This follows at once from Art 13, for it was there shown that $b = \frac{p}{n}$, and, as in this case, $p = 0$, it follows that $b = 0$ In this case, the graph cannot be constructed as in the two preceding examples. Since the line passes through the origin O, Fig. 17, it is only necessary to determine another point. This is done by assuming any convenient value for x and solving for y Making $x = \frac{1}{2}$, the equation becomes $4y - 4 = 0$, whence $y = 1$ Laying off $OM = \frac{1}{2}$, and the ordinate $MP = 1$, the line A_2B_2, drawn through O and P, is the required graph

EXAMPLES FOR PRACTICE

1 Draw the graphs of the equations (a) $8x - 8y + 24 = 0$. (b) $6x + 4y = 0$ (c) $5x + 4y = -20$. (d) $12y - 10x + 30 = 0$.

Ans $\left\{\begin{array}{l}\text{The graphs are shown, respectively, at } A_1B_1, \\ A_2B_2,\ A_3B_3,\ A_4B_4 \text{ in Fig. 18}\end{array}\right.$

2 Find the angle that the graph of the equation $4x + 5y - 4 = 0$ makes with the x axis. Ans. 141° 20′ 20″

3 Given the equation $x + 4y = 20$, find (*a*) the intercept of the graph on the y axis, (*b*) the angle that the graph makes with the x axis (Give seconds in angle to nearest multiple of 10.)

Ans $\begin{cases} (a) & 5 \\ (b) & 165° 57' 50'' \end{cases}$

APPLICATIONS

16. Reactions on a Beam.—Let a beam AB, Fig. 19, resting on the supports A and B, carry a movable load W. Let the distance of the load from the left support at any instant be x, and let the length of the beam be denoted by l. For this position of the load, the reaction R_1 at A is obtained by taking moments about B; thus,

$$R_1 l - W(l - x) = 0;$$

whence $$R_1 = W - \frac{W}{l} \times x \quad (1)$$

Since the load W moves, x is a variable, and so is R_1. Equation (1) gives R_1 as a function of x. As that equation is of the first degree, it can be represented by a straight line, R_1 being used instead of the y used in previous articles. A convenient and usual way of drawing the graph is as follows: The origin O is taken directly under A, and the axis OX is drawn parallel to AB; the axis OY is drawn through O. Equation (1) shows that the y intercept is W. Therefore, laying off, to any convenient scale, along OY the distance OI to represent W, one point I of the graph is obtained. The equation also shows that the x intercept (found by making

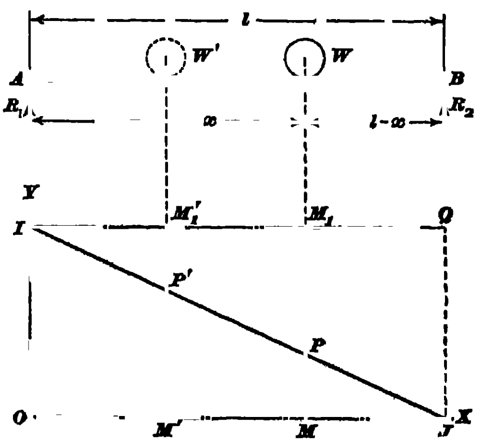


FIG. 19

$R_1 = 0$) is I. Therefore, projecting the point B on OX at J, the point J of the graph is found, and the graph is the straight line IJ. The ordinate MP represents the left reaction when the load is at W; the ordinate $M'P'$ represents the left reaction when the load is at W'; etc. If IQ is drawn parallel to OX, the ordinates M_1P, $M_1'P'$, etc. will represent the corresponding values of the right reaction, since $R_1 + R_2 = W = OI$. If, for example, the load is 2,000 pounds, and a scale of 1,000 pounds to the inch is used, OI should be made 2 inches; and, if the ordinate $M'P'$ is found to measure $1\frac{1}{2}$ inches, the reaction R_1, when the load is at W', is $1{,}000 \times 1\frac{1}{2} = 1{,}500$ pounds.

17. Pressure on the Back of a Dam.—Let OM_1, Fig. 20, be the back or inner face, supposed to be vertical, of a dam, the water reaching to the top O From hydrostatics it is known that the pressure p per square foot, at any point M whose depth below the surface is x feet, is given by the formula

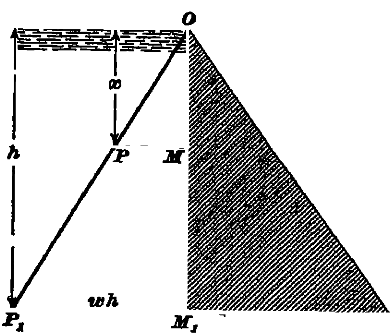


FIG 20

$$p = wx$$

in which w is the weight of 1 cubic foot of water. As this is an equation of the first degree, it can be represented by a straight line. Taking OM_1 as the axis of x and O as the origin, the latter point is a point in the graph, since there is no intercept. Making $x = h$, the formula gives $p = wh$, which is the pressure at M_1 Laying off, horizontally, $M_1P_1 = wh$, the line OP_1 is the required graph. The intensity of pressure at any point M is given by the ordinate MP.

If, for instance, the height h is 24 feet, and w is taken equal to 62.5 pounds, $wh = 62\ 5 \times 24 = 1{,}500$ pounds per square foot. If a scale of 500 pounds per square foot to the inch is used, M_1P_1 should be made $1{,}500 \div 500 = 3$ inches;

and, if MP measures $1\frac{1}{4}$ inches, the pressure at M is

$$500 \times 1\tfrac{1}{4} = 625 \text{ pounds per square foot.}$$

It will be observed that here p is used instead of y, that the x axis is vertical; and that positive values of x are counted downwards, and positive values of p toward the left It is often necessary to make such changes in notation, so as to adapt the construction to given conditions. The general principles and methods, however, remain the same.

THE PARABOLA

18. Definitions.—A **parabola**, Fig. 21, is a curve such that, if any point P on it is taken, the distance PF of that point from a fixed point F is equal to its distance PN from a fixed line DE. The fixed point F is called the **focus**; the fixed line DE, the **directrix**. The line N_0X passing through the focus and perpendicular to the directrix is called the **axis** of the curve, and the point O where the curve crosses the axis is called the **vertex**. Twice the distance FN_0 from the focus to the directrix is called the **parameter**, and is denoted by $2p$; so that $2p = 2N_0F$, and $p = N_0F$.

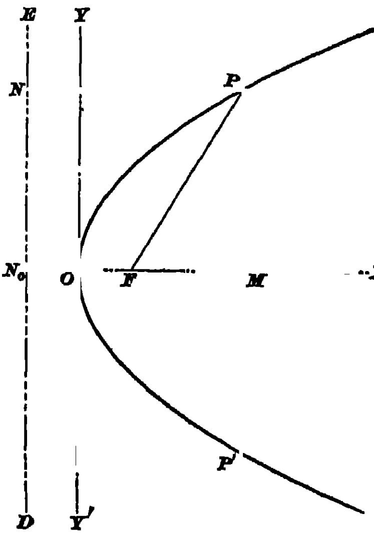


FIG. 21

The curve is symmetrical with respect to the axis; that is, to every point P on one side of the axis there corresponds another point P' on the other side, at the same distance from, and on the same perpendicular to, the axis. Any line, as PP', perpendicular to the axis and bounded at its two ends by the curve is called a **double ordinate**.

19. Equation of the Parabola Referred to the Axis and Vertex.—Let P, Fig. 21, be any point on the curve, and $OM = x$ and $MP = y$ its coordinates, the axes being the axis OX of the parabola and the perpendicular OY at O Let the parameter be $2p$. Then, the distance FN_0 from the focus F to the directrix DE is equal to p, and $OF = \frac{1}{2}p$, since $ON_0 = OF$, the point O of the curve being, according to the definition of a parabola, equally distant from the directrix DE and the focus F. According to the same definition, we have

$$PF = PN$$

and, therefore,

$$PF^2 = PN^2 \qquad (1)$$

Now,

$$PF^2 = PM^2 + FM^2 = y^2 + (OM - OF)^2$$
$$= y^2 + (x - \tfrac{1}{2}p)^2 = y^2 + x^2 - px + \frac{p^2}{4}$$

Also,

$$PN^2 = N_0M^2 = (OM + N_0O)^2 = (x + \tfrac{1}{2}p)^2 = x^2 + px + \frac{p^2}{4}$$

Substituting in equation (1) these values of PF^2 and PN^2,

$$y^2 + x^2 - px + \frac{p^2}{4} = x^2 + px + \frac{p^2}{4},$$

whence
$$y^2 = 2px$$
which is the required equation.

20. Let y_1 and y_2 be two ordinates corresponding to the abscissas x_1 and x_2 Then, since the equation applies to all points, $y_1^2 = 2px_1$, $y_2^2 = 2px_2$, whence

$$\frac{x_1}{x_2} = \frac{y_1^2}{y_2^2}$$

That is, *in any parabola, any two coordinates parallel to the axis of the curve are to each other as the squares of the corresponding coordinates perpendicular to the axis.*

21. It is important that the equation of the parabola should be so mastered that it can be applied to parabolas in different positions, and when the notation is different from the one here used. It should be borne in mind that in the

general equation, the coordinate to be squared is that perpendicular to the axis of the parabola. In Fig. 22 is represented a parabola with the axis vertical, coinciding with the y axis. In this case, the equation should be written

$$x^2 = 2py$$

Fig. 23 represents a vertical parabola with its axis down-

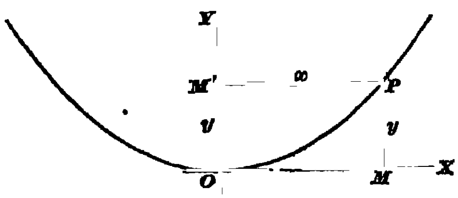


FIG. 22

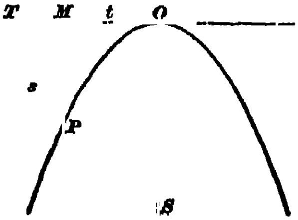


FIG 23

wards The coordinate axes are denoted by OS and OT, as shown, and the coordinates of any point P by s and t, s being positive downwards, and t positive toward the left Under such conditions, the equation of the curve should be written

$$t^2 = 2ps$$

22. Problem I.—*To find the parameter and equation of a parabola, when a double ordinate and the corresponding abscissa are given.*

This is the usual way in which the parabola occurs in practice. Thus, in road construction, in which the cross-

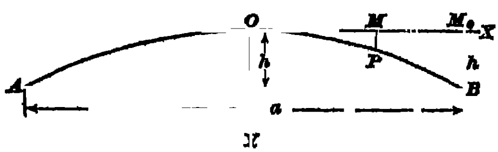


FIG. 24

section of a road is often made parabolic, the width a of the road, Fig. 24, and the height h of the center above the ends A and B are given. In order to determine the fall of the cross-section at different distances from the center, it is necessary, or at least advisable, to determine the equation and parameter of the curve AOB.

As usual, the parameter will be denoted by $2p$. At the point M_0, for which $x = \frac{a}{2}$, and $y = h$, we have, since here the x coordinate is perpendicular to the axis,

$$\left(\frac{a}{2}\right)^2 = 2ph;$$

whence

$$2p = \frac{a^2}{4h}$$

The general equation of the parabola is, therefore,

$$x^2 = \frac{a^2}{4h} \times y$$

In practice, points on the curve are determined by assuming values for x, and computing the corresponding values of y. The equation may, therefore, be more conveniently written in the form

$$y = \frac{4hx^2}{a^2} = h \times \left(\frac{x}{\frac{1}{2}a}\right)^2$$

Let OM_0 be divided into any convenient number of equal

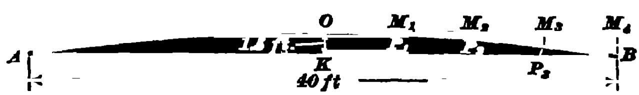


FIG 25

parts, say n, and give to x successive values corresponding to the points of division; thus, $x = \frac{OM_0}{n}$, $x = 2\frac{OM_0}{n}$, $x = 3\frac{OM_0}{n}$, etc.; or $x = \frac{\frac{1}{2}a}{n}$, $x = 2\frac{\frac{1}{2}a}{n}$, $x = 3\frac{\frac{1}{2}a}{n}$, etc. Then, the corresponding values of y become

$$h \times \left(\frac{\frac{1}{2}a \div n}{\frac{1}{2}a}\right)^2 = h \times \left(\frac{1}{n}\right)^2 = \frac{h}{n^2} \times 1$$

$$h \times \left(\frac{2 \times \frac{1}{2}a - n}{\frac{1}{2}a}\right)^2 = h \times \left(\frac{2}{n}\right)^2 = \frac{h}{n^2} \times 2^2$$

$$h \times \left(\frac{3 \times \frac{1}{2}a \div n}{\frac{1}{2}a}\right)^2 = h \times \left(\frac{3}{n}\right)^2 = \frac{h}{n^2} \times 3^2, \text{ etc.}$$

Thus, if OM_0 is divided into ten equal parts, and OM contains six of those parts,

$$y = MP = \frac{h}{10^2} \times 6^2 = \frac{h}{100} \times 36$$

EXAMPLE —Given the double ordinate $AB = 40$ feet, Fig 25, and the corresponding abscissa $OK = 1\ 5$ feet, it is desired to find points on the curve at intervals of 5 feet on each side of O

SOLUTION —Draw OM_4 parallel to AB, and BM_4 parallel to KO Since the points are to be located every 5 ft from O, and $OM_4 = 20$ ft, the latter line should be divided into four equal parts OM_1, M_1M_2, etc Here, $\frac{1}{2}a = \frac{1}{2} \times 40 = 20$, $n = \frac{20}{5} = 4$, and $h = 1\ 5$. Therefore, $\frac{h}{n^2} = \frac{1\ 5}{16} = 094$, nearly, and the values of y are

at M_1, $M_1P_1 = 094 \times 1 = 09$ ft, nearly,
at M_2, $M_2P_2 = 094 \times 2^2 = 094 \times 4 = 38$ ft, nearly,
at M_3, $M_3P_3 = .094 \times 3^2 = .094 \times 9 = .85$ ft., nearly,
at M_4, $M_4B = 1\ 5$ ft

23. Problem II.—*To construct a parabola when a double ordinate and the corresponding abscissa are given.*

In practice, what is usually required is to locate points of the curve on the ground, as in the case of road and street construction, and in railroad curves. Under such circumstances, the ordinates are calculated as explained in Art. 22; if desirable, they may be platted, and the curve drawn through the points thus determined. A purely graphic method of constructing the curve is explained in *Geometrical Drawing*. Another method, which is often convenient in the drafting room, is as follows:

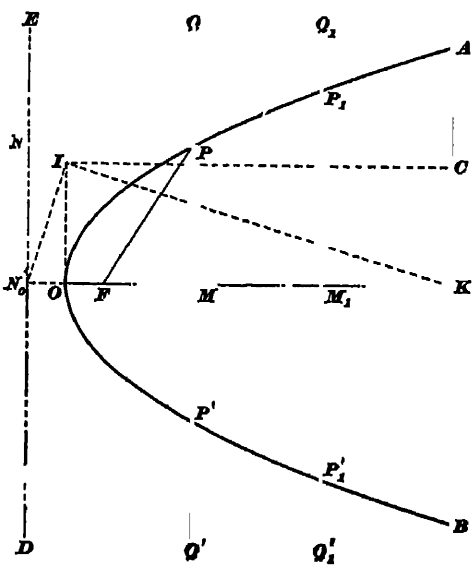


FIG 26

Let AB, Fig. 26, be the given double ordinate, and OK the corresponding abscissa. Bisect KA at C and draw CI parallel to KO and OI perpendicular to KO. Draw IK,

and IN_0 perpendicular to it, meeting the axis produced at N_0. The line DE, drawn through N_0 perpendicular to OK produced, is the directrix It is shown here for the purpose of explanation, but it is not necessary to draw it. Lay off $OF = ON_0$. The point F is the focus. From N_0, lay off along the axis any convenient distances, as N_0M, N_0M_1, etc , and at the points M, M_1, etc. draw indefinite perpendiculars QQ', Q_1Q_1', to the axis. From F as a center, and with a radius equal to N_0M, describe an arc, cutting QQ' at P and P': these are points of the curve. Likewise, the points P_1 and P_1' are the intersections of Q_1Q_1' with an arc described from F with a radius equal to N_0M_1 Other points may be determined in a similar manner, and the curve drawn through them.

It is usually more convenient to find the points N_0 and F by calculation, instead of by the geometrical construction described above. The parameter $2p$ is computed as explained in Art. 22, and then ON_0 and OF are laid off each equal to $\frac{1}{2}p$.

NOTE —The correctness of the preceding construction will now be shown It is not necessary for the student to study the following demonstration, but he is advised to read it carefully, as it is a good exercise As explained in Art 22, $2p = \frac{a^2}{4h}$ In Fig 26, $a = 2AK$ and $h = OK$. Therefore,

$$2p = \frac{(2AK)^2}{4OK} = \frac{AK^2}{OK};$$

whence
$$\frac{p}{2} = \frac{\frac{AK^2}{4}}{OK} = \frac{(\frac{1}{2}AK)^2}{OK} \qquad (1)$$

In the right triangle N_0IK, the perpendicular OI, which is equal to $\frac{1}{2}AK$, is a mean proportional between OK and ON_0, that is,

$$(\tfrac{1}{2}AK)^2 = ON_0 \times OK,$$

whence
$$ON_0 = \frac{(\frac{1}{2}AK)^2}{OK} = \frac{p}{2} \text{ [by equation (1)]}$$

Therefore, ON_0 is the distance from the vertex to the directrix (Art 18)

The point P was so determined that $FP = N_0M = PN$. That point is, therefore, at the same distance from the focus as from the directrix, and, according to the definition of a parabola, must be a point of the curve The same reasoning applies to the points P', P_1, etc.

APPLICATIONS

24. Projectiles.—In the most general sense of the term, a **projectile** is any body thrown into the air. The velocity with which a projectile is thrown is called the **initial velocity**, or **velocity of projection.** Here, only projectiles thrown horizontally will be considered, and the resistance of the air will be neglected.

Let a projectile be thrown horizontally in the direction OX from a point O, Fig. 27, with a velocity v. It is required to determine the path OA of the projectile. The horizontal line OX and the vertical line OY will be taken as axes of coordinates, y being positive downwards. Were the projectile not acted on by gravity, it would describe, in any time t, a space $OM = vt$, which will be denoted by x. If it were not thrown at all, and were allowed to fall freely from O during the time t, it would fall through a distance $OG = \frac{1}{2}gt^2$, denoting, as usual, by g the acceleration due to gravity. This distance OG will be denoted by y. When the projectile, after being thrown along OX with the velocity v, is acted on by gravity, its position P, at the end of the time t, will be such that MP will be equal and parallel to OG. We have, therefore,

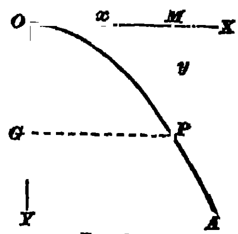


FIG. 27

$$x = vt$$

and $$y = \frac{1}{2}gt^2.$$

From the first of these two equations is found $t^2 = \frac{x^2}{v^2}$. Substituting this value in the second equation, there results

$$y = \frac{1}{2}g\frac{x^2}{v^2};$$

whence $$x^2 = \frac{2v^2}{g}y,$$

which is the equation of a parabola with its vertex at O and axis vertical. This parabola is, therefore, the path followed by the projectile. Its parameter is $\frac{2v^2}{g}$.

25. Jet of Water Issuing From a Small Orifice. One of the most important applications of the theory of projectiles is to the determination of the velocity of water flowing from a small orifice. This determination is of much importance in hydraulics. In Fig. 28 is represented a tank T from which water flows through a small orifice O. The water issues horizontally with a velocity v (to be determined), and each particle, being under the same conditions as a projectile thrown horizontally with the same velocity, describes a parabola. As the jet is narrow, it may be treated as a whole as a parabolic arc OJ. A horizontal string OX is stretched from O, and any distance OM_1 is measured. From M_1 another string carrying a heavy weight is suspended, and the distance M_1P_1 is measured. Let $OM_1 = x_1$, $M_1P_1 = y_1$. Then,

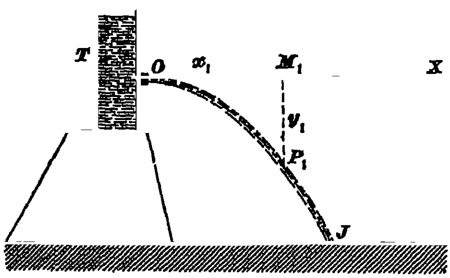


FIG 28

$$x_1^2 = \frac{2v^2}{g} y_1;$$

whence
$$v = \sqrt{\frac{g x_1^2}{2 y_1}}$$

EXAMPLE.—What was the velocity of a jet for which the measured distances x_1 and y_1 were, respectively, 6 and 5 feet?

SOLUTION —Substituting in the formula the given values of x_1 and y_1, and 32 16 for g,

$$v = \sqrt{\frac{32\ 16 \times 6^2}{2 \times 5}} = \sqrt{3\ 216 \times 36} = 10\ 76 \text{ ft. per sec} \quad \text{Ans}$$

26. Moment in a Beam.—Let a beam AB, Fig. 29, of length l, carry a movable load W. When the load is at a distance x from the center of the beam, the left reaction R_1 is found by taking moments about B; thus,

$$R_1 l = W(\tfrac{1}{2} l - x);$$

whence
$$R_1 = \frac{W}{l}(\tfrac{1}{2} l - x)$$

The moment of R_1 about W is called the **bending moment** at W. Denoting it by M, we have

$$M = R_1(\tfrac{1}{2}l + x) = \frac{W}{l}(\tfrac{1}{2}l - x)(\tfrac{1}{2}l + x)$$

$$= \frac{W}{l}\left(\frac{l^2}{4} - x^2\right) = \frac{Wl}{4} - \frac{W}{l}x^2 \qquad (1)$$

The bending moment at the center, which will be denoted by M_0, is obtained by making $x = 0$, which gives

$$M_0 = \frac{Wl}{4}$$

Equation (1) may be written

$$M = M_0 - \frac{W}{l}x^2,$$

whence

$$x^2 = \frac{l}{W}(M_0 - M)$$

If $M_0 - M$ is denoted by y, equation (1) may be written

$$x^2 = \frac{l}{W}y$$

which is the equation of a parabola whose axis is perpendicular to AB, and whose parameter is $\frac{l}{W}$. The curve

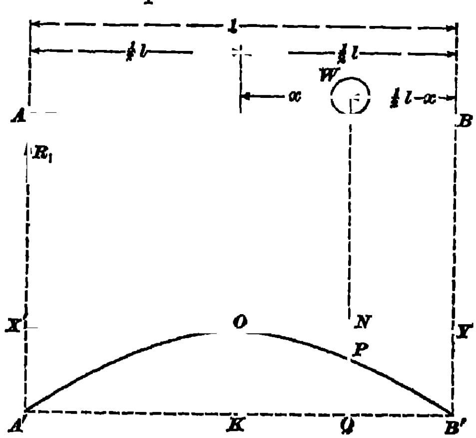


FIG. 29

is shown at $A'OB'$. It may be constructed by points, assuming values of x and finding the corresponding values of M and $M_0 - M$; or by the method of Art. **23**, noticing that to the double ordinate $A'B'$, or l, corresponds a value of y ($= KO$) equal to M_0, since, when $x = \frac{1}{2}l$, $M = 0$, and $y = M_0 - M = M_0$. Having drawn the curve, the bending moment, when the weight is at any point W, is found by projecting W on $A'B'$, and measuring the distance QP, which is equal to the moment at W, to the scale by which KO represents M_0. For the ordinate NP represents y, or $M_0 - M$, and, therefore,

$$M_0 - M = NP,\ M = M_0 - NP = QN - NP = QP$$